Numerical Methods

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Contents

Course information:

- **Course name** : Math II (Numerical Methods)
- **Instructor** : Dr. Sachchidanand Prasad
- **Course webpage** : Link to the course website
- **References** :

[1] *Advanced Engineering Mathematics*, by Erwin Kreyszig.

[2] *Higher Engineering Mathematics*, by B.S. Grewal

1 Solution of Polynomial and Transcendental Equations

In numerical analysis, the solution of polynomial and transcendental equations is a fundamental topic with widespread applications in various fields of science and engineering. These equations, which are often too complex to solve analytically, require iterative numerical methods to find their roots.

A polynomial of degree n is of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad \text{where } a_n \neq 0.$$

The polynomial $f(x) = 0$ is called an *algebraic equation* of degree n. If $f(x)$ contains other functions in its coefficients, like trigonometric functions or the exponential function, then $f(x) = 0$ is referred to as a *transcendental equation*. For example,

$$7x^3 - 3x^2 + 2 = 0$$

is an algebraic equation, whereas

$$\sin x - e^x = 0$$

is a transcendental equation.

If there exists a real number α such that $f(\alpha) = 0$, then we say α is a real root (or zero) of the equation $f(x) = 0$. The main goal of this section is to find a real root of the equation $f(x) = 0$.

The root of an equation can be found either analytically or numerically. Not all equations can be solved analytically. For example, the zeros of the equation

$$x^2 - 3x + 2 = 0$$

can be found using the quadratic formula. Similarly, the roots of cubic and quartic polynomials can be determined analytically. However, for more complex equations, such as

$$\sin x - e^x = 0,$$

we must rely on iterative or numerical methods. In iterative methods, the root is approximated by starting with an initial guess (or initial approximation) and gradually improving this guess to achieve a more accurate approximation of the root.

In this section, we will explore various techniques for solving these types of equations, including:

- **Bisection Method**: A simple and robust method that repeatedly bisects an interval and selects the subinterval in which the root lies.
- **Regula Falsi Method (False Position Method)**: A method that improves upon the bisection method by interpolating a straight line between two points on the function curve.
- **Newton-Raphson Method**: A powerful and fast method that uses the derivative of the function to iteratively approximate the root.

Each method has its own advantages, limitations, and applicability depending on the nature of the equation and the desired accuracy of the solution.

1.1 Bisection Method

The bisection method is a simple and robust numerical technique for finding the roots of a continuous function $f(x)$ within a specified interval $[a, b]$. The method is particularly useful when you know that the function changes sign within the interval, meaning there is at least one root x such that $f(x) = 0$.

1.1.1 Basic Idea

The bisection method relies on the *Intermediate Value Theorem*, which states that *if a continuous function changes sign over an interval, then there must be at least one root within that interval.* The method works by repeatedly halving the interval $[a, b]$ and selecting the subinterval where the sign change occurs (see Figure 1).

The algorithm for the bisection method is as follows.

Step 1: Choose a and b such that $f(a) \cdot f(b) < 0$, that is, $f(a)$ and $f(b)$ are of different signs.

Step 2: Calculate the first iteration of the numerical solution x_1 by

$$x_1 = \frac{a+b}{2}.$$

Step 3: Now check whether $f(x_1) \cdot f(a) < 0$ or $f(x_1) \cdot f(b) < 0$.

- If $f(x_1) \cdot f(a) < 0$, then for the next iteration replace b by x_1. Now new $a = a$ and $b = x_1$.
- If $f(x_1) \cdot f(b) < 0$, then for the next iteration replace a by x_1. Now new $a = x_1$ and $b = b$.

Step 3: Repeat the process until the interval is sufficiently small, at which point the midpoint is taken as the approximate root.

1.1.2 Convergence

The bisection method guarantees convergence to a root as long as the function is continuous and the initial interval contains a root. The convergence is linear, meaning that the error is reduced by about half with each iteration. While the method is not as fast as some other root-finding algorithms, its reliability makes it a valuable tool, especially when the behavior of the function is not well understood.

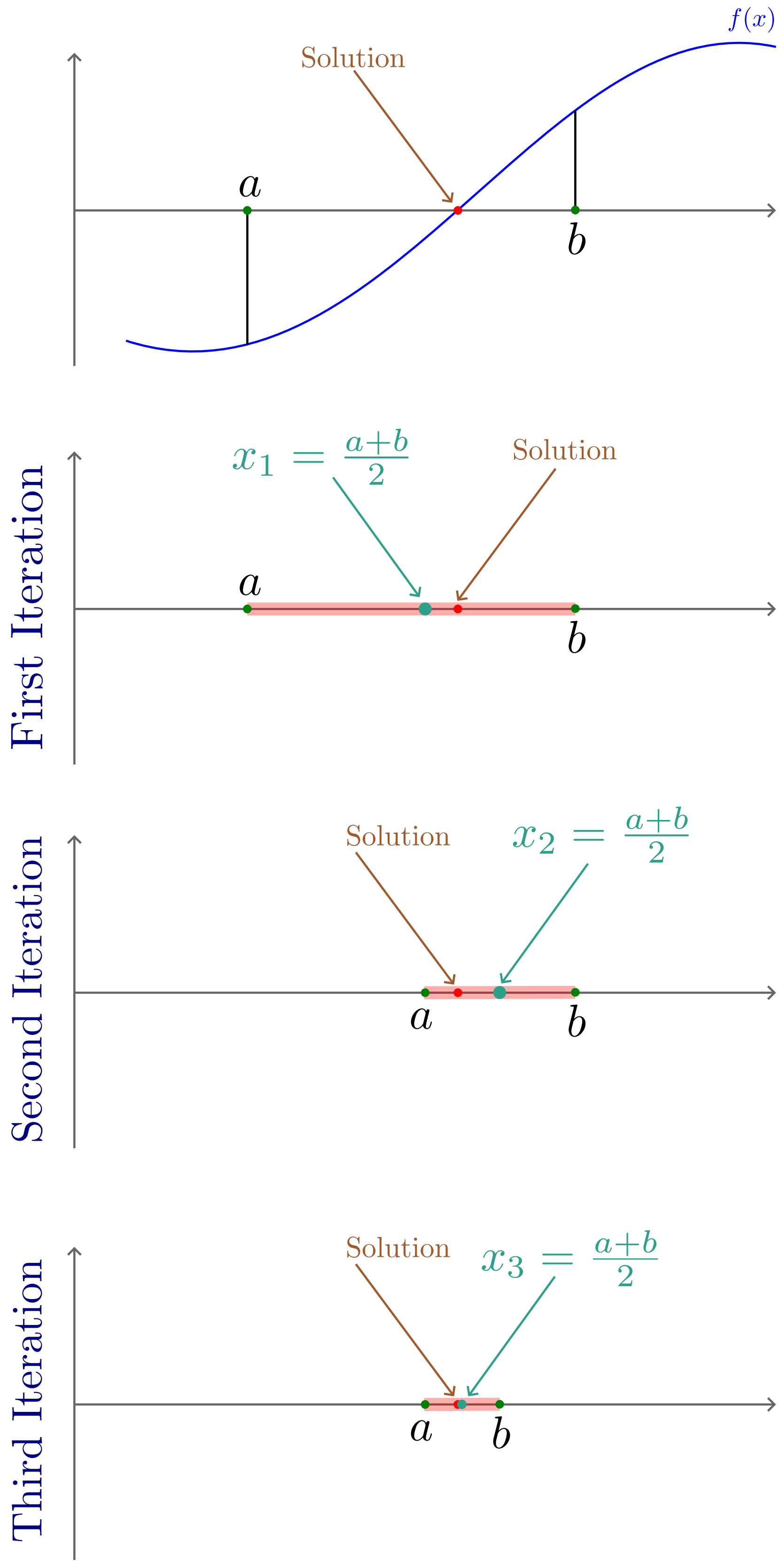

Figure 1: Bisection Method

Example 1.1

Use bisection method to find the real root of the equation $f(x) = x^3 - x - 1 = 0$. Show five iterations.

Solution

Here we have

$$f(1) = -1 \text{ and } f(2) = 5,$$

so we can use bisection method with $a = 1$ and $b = 2$.

- **First iteration:**

$$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5.$$

Note that $f(1.5) = 0.88$, which is positive. So, for the next iteration we have

$$a = 1 \text{ and } b = 1.5.$$

- **Second iteration:**

$$x_2 = \frac{a+b}{2} = \frac{1+1.5}{2} = 1.25.$$

Since $f(1.25) = -0.3$, we take $a = 1.25$ and $b = 1.5$ for the next iteration.

- **Third iteration:**

$$x_3 = \frac{a+b}{2} = \frac{1.25+1.5}{2} = 1.375.$$

Since $f(1.375) > 0$, we take $a = 1.25$ and $b = 1.375$ for the next iteration.

- **Fourth iteration:**

$$x_4 = \frac{a+b}{2} = \frac{1.25+1.375}{2} = 1.3125.$$

Since $f(1.375) < 0$, we take $a = 1.3125$ and $b = 1.375$ for the next iteration.

- **Fifth iteration:**

$$x_4 = \frac{a+b}{2} = \frac{1.3125+1.375}{2} = 1.34375.$$

Since $f(1.34375) > 0$, we take $a = 1.3125$ and $b = 1.34375$ for the next iteration.

In the following table, we can give more iterations.

a	b	$f(a)$	$f(b)$	root
1.0000	2.0000	-1.0000	5.0000	1.5000
1.0000	1.5000	-1.0000	0.8750	1.2500
1.2500	1.5000	-0.2969	0.8750	1.3750
1.2500	1.3750	-0.2969	0.2246	1.3125
1.3125	1.3750	-0.0515	0.2246	1.3438
1.3125	1.3438	-0.0515	0.0826	1.3281
1.3125	1.3281	-0.0515	0.0146	1.3203
1.3203	1.3281	-0.0187	0.0146	1.3242
1.3242	1.3281	-0.0021	0.0146	1.3262
1.3242	1.3262	-0.0021	0.0062	1.3252
1.3242	1.3252	-0.0021	0.0020	1.3247
1.3247	1.3252	-0.0000	0.0020	1.3250
1.3247	1.3250	-0.0000	0.0010	1.3248
1.3247	1.3248	-0.0000	0.0005	1.3248
1.3247	1.3248	-0.0000	0.0002	1.3247
1.3247	1.3247	-0.0000	0.0001	1.3247

1.2 Regula Falsi Method

The Regula Falsi Method, also known as the False Position Method, is an iterative numerical technique for finding the roots of a continuous function $f(x)$. Like the bisection method, it relies on the principle that a function that changes sign over an interval $[a, b]$ must have a root within that interval. However, the Regula Falsi Method uses a more sophisticated approach to choose the next approximation, leading to potentially faster convergence.

1.2.1 Basic Idea

The method improves upon the bisection method by using a linear interpolation technique to estimate the root. Instead of simply taking the midpoint of the interval, the method draws a straight line (or "secant" or "chord") between the points $(a, f(a))$ and $(b, f(b))$, and takes the x-intercept of this line as the next approximation.

At first we choose a and b such that the root lies between a and b, that is, $f(a) \cdot f(b) < 0$. Next we consider the chord joining $f(a)$ and $f(b)$ (see Figure 2). The numerical solution will be the x-coordinate of the point where the chord intersects the x-axis.

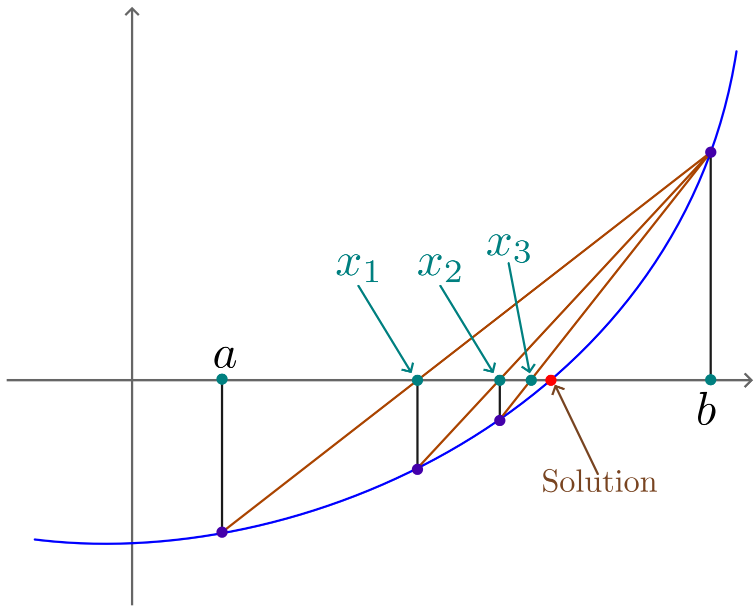

Figure 2: Regula Falsi Method

Let us derive the value of x_1. The equation of the chord (line) joining $(a, f(a))$ and $(b, f(b))$ is given by

$$\frac{y - f(b)}{f(b) - f(a)} = \frac{x - b}{b - a}.$$

We need to determine when does this chord intersect the x-axis. When the line intersects the

x-axis, its y-coordinate is 0. So,

$$\frac{0-f(b)}{f(b)-f(a)} = \frac{x_1-b}{b-a} \implies (x_1-b)(f(b)-f(a)) = -f(b)(b-a)$$

$$\implies x_1 - b = \frac{-f(b)(b-a)}{f(b)-f(a)}$$

$$\implies x_1 = b - \frac{f(b)(b-a)}{f(b)-f(a)}$$

$$= \frac{bf(b)-bf(a)-bf(b)+af(b)}{f(b)-f(a)}$$

$$= \frac{af(b)-bf(a)}{f(b)-f(a)}.$$

Now we check which one of $f(a)\cdot f(x_1)$ and $f(b)\cdot f(x_1)$ is negative. For example, if $f(a)\cdot f(x_1) < 0$, then for the next iteration we replace b by x_1 and a remains unchanged. The regula falsi algorithm is given below.

Step 1: Choose a and b such that $f(a)\cdot f(b) < 0$, that is, $f(a)$ and $f(b)$ are of different signs.

Step 2: Calculate the first iteration of the numerical solution x_1 by

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}.$$

Step 3: Now check whether $f(x_1)\cdot f(a) < 0$ or $f(x_1)\cdot f(b) < 0$.

- If $f(x_1)\cdot f(a) < 0$, then for the next iteration replace b by x_1. Now new $a = a$ and $b = x_1$.
- If $f(x_1)\cdot f(b) < 0$, then for the next iteration replace a by x_1. Now new $a = x_1$ and $b = b$.

Step 4: Repeat the process until the interval is sufficiently small, at which the next iteration is taken as the approximate root.

1.2.2 Convergence

The method typically converges faster than the bisection method, particularly when the function is nearly linear in the vicinity of the root. However, it is not guaranteed to converge faster in all cases. In some situations, the method can become ”stuck” and converge slowly if the secant lines consistently favor one side of the interval.

Example 1.2

The equation $2x = \log_{10} x + 7$ has a root between 3 and 4. Find this root, correct to three decimal places, by regula falsi method.

Solution

Note that

$$f(3) = -1.4771 \text{ and } f(4) = 0.3979,$$

so a root lie in the interval $[3, 4]$. Applying the regula falsi method, we have the following iterations.

- **First iteration:**

$$\begin{aligned} x_1 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{3 \times 0.3979 - 4 \times (-1.4771)}{0.3979 + 1.4771} \\ &= 3.7878. \end{aligned}$$

  Note that $f(3.7878) < 0$, so for the next iteration we have

$$a = 3.7878 \text{ and } b = 4.$$

- **Second iteration:**

$$\begin{aligned} x_2 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{3.7878 \times (0.3979) - 4 \times (-0.0028)}{0.3979 + 0.0028} \\ &= 3.7893. \end{aligned}$$

  Since $f(3.7893) < 0$, we take $a = 3.7893$ and $b = 4$ for the next iteration.

- **Third iteration:**

$$\begin{aligned} x_3 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= 3.7893. \end{aligned}$$

  Thus, the root between 3 and 4 is 3.7893, which is correct to 3 decimal places.

Here is the table

a	b	f(a)	f(b)	root
3.0000	4.0000	-1.4771	0.3979	3.7878
3.7878	4.0000	-0.0028	0.3979	3.7893
3.7893	4.0000	-0.0000	0.3979	3.7893
3.7893	4.0000	-0.0000	0.3979	3.7893
3.7893	4.0000	-0.0000	0.3979	3.7893
3.7893	4.0000	-0.0000	0.3979	3.7893

1.3 Newton Raphson Method

The Newton-Raphson method is a powerful and widely-used iterative technique for finding the roots of a real-valued function $f(x)$. Unlike the Bisection and Regula Falsi methods, which do not require the computation of derivatives, the Newton-Raphson method leverages the derivative of the function to achieve rapid convergence to the root. This method is particularly effective when an initial approximation to the root is already known.

1.3.1 Basic Idea

It is based on the idea of linear approximation. Starting with an initial guess x_0 for the root of the equation $f(x) = 0$, the method uses the tangent line to the curve of $f(x)$ at x_0 to generate a better approximation x_1. This process is repeated, producing a sequence of values $x_0, x_1, x_2, \ldots,$, that converge to the actual root (look at the Figure 3).

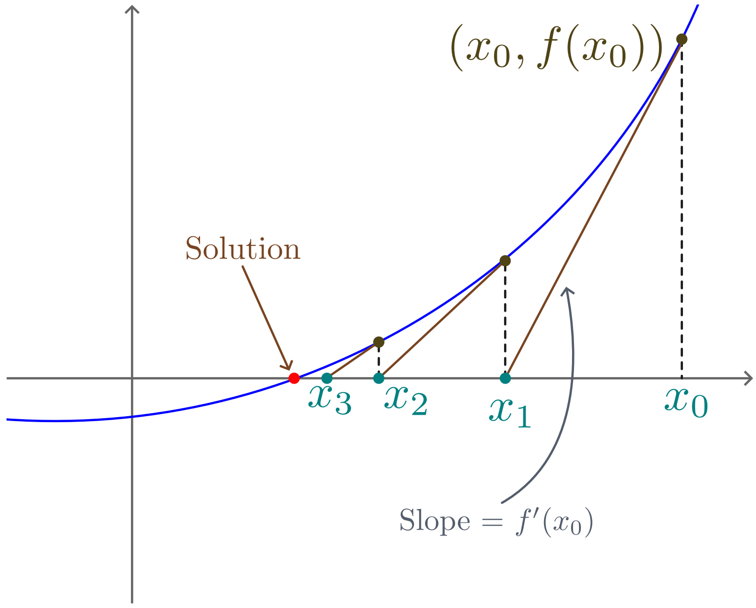

Figure 3: Newton Raphson Method

We now derive an iteration scheme for the Newton's method. The equation of the tangent line passing through $(x_0, f(x_0))$ will be

$$y - f(x_0) = f'(x_0)(x - x_0).$$

This line intersects the x-axis at x_1, which can be find by putting $y = 0$. So,

$$\begin{aligned} 0 - f(x_0) = f'(x_0)(x_1 - x_0) &\implies x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} \\ &\implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \end{aligned}$$

Note that the above computation will work only if $f'(x_0) \neq 0$. Thus, in the Newton Raphson method, we need to choose x_0 such that $f'(x_0) \neq 0$. Therefore, the iterative formula for the Newton-Raphson method is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots.$$

The algorithm is discussed below.

Step 1: Choose a point x_0 as the initial guess of the solution. Make sure that $f'(x_0) \neq 0$.

Step 2: For $n = 0, 1, 2, \ldots$, until the error is smaller than a specified value, calculate x_{n+1} by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

1.3.2 Convergence

One of the main advantages of the Newton-Raphson method is its rapid convergence, especially when the initial guess is close to the actual root. Under ideal conditions, the method exhibits quadratic convergence, meaning that the number of correct digits in the approximation roughly doubles with each iteration. However, the method's performance can be hindered if the derivative $f'(x)$ is zero or very small near the root, leading to potential convergence issues or even divergence.

Example 1.3

Find a root of the equation $x\sin x = -\cos x$ correct to four decimal places.

Solution

The given equation is

$$x\sin x = -\cos x \implies f(x) = x\sin x + \cos x.$$

The derivative will be

$$f'(x) = \sin x + x\cos x - \sin x = x\cos x.$$

From the Newton Raphson iteration scheme, we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0 \\ &= x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}, \quad n \geq 1. \end{aligned}$$

We start with an initial guess as $x_0 = \pi$.

- **First iteration:**

$$\begin{aligned} x_1 &= x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0} \\ &= \pi - \frac{0-1}{-\pi} \\ &= \pi - \frac{1}{\pi} = 2.823283. \end{aligned}$$

- **Second iteration:**

$$\begin{aligned} x_2 &= x_1 - \frac{x_1 \sin x_1 + \cos x_1}{x_1 \cos x_1} \\ &= 2.798600. \end{aligned}$$

- **Third iteration:**

$$
\begin{aligned}
x_3 &= x_2 - \frac{x_2 \sin x_2 + \cos x_2}{x_2 \cos x_2} \\
&= 2.798386.
\end{aligned}
$$

- **Fourth iteration:**

$$
\begin{aligned}
x_4 &= x_3 - \frac{x_3 \sin x_3 + \cos x_3}{x_3 \cos x_3} \\
&= 2.798386.
\end{aligned}
$$

The following table summarizes the computation.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	π	-1.000000	-π	2.823283
1	2.823283	-0.066186	-2.681457	2.798600
2	2.798600	-0.000564	-2.635588	2.798386
3	2.798386	-0.000000	-2.635185	2.798386

Therefore, the solution of the given equation correct to four decimal places is 2.798386.