MATRIX GROUPS

(MTH565)

Quiz 6

Thursday, 16th October 2025

Name: ______________________________

Roll Number: ______________________________

Obtained Marks: ______________________ **/10**

EXAMINATION INSTRUCTIONS

1. This is an **Open Book Examination**.
2. You can discuss with your friends but make sure write your own script. Answer all questions.
3. Show all necessary working steps clearly and legibly.
4. State any theorems or results used. Only results discussed in lectures may be used without proof.
5. The total point for the problems is 12, but the maximum obtainable score is 10.
6. **Duration:** 9 hours (12:00-9:00 PM).
7. I will assume your honesty that you will stop at 9:00 PM and keep your answer script with you until I take it.

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Good Luck!

Problem Set

The set of problems will describe an isomorphism between the quotient $SU(2)/\{\pm I\} \cong SO(3)$ which was a problem in the Homework 5 (Problem 2 (iii)).

Problem 1

The n-sphere is defined as

$$\mathbb{S}^n := \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

Show that S^3 is a subgroup of the multiplicative group of quaternions $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$.

1.5

Problem 2

Prove that $SU(2) \cong S^3$ as a group.

1.5

Problem 3

Let $\text{Im}\,\mathbb{H} = \{x\mathbf{i} + y\mathbf{j} + z\mathbf{k} : x, y, z \in \mathbb{R}\}$ be the set of pure quaternions, which we identify with $\mathbb{R}^3$. For $q \in S^3$ and $v \in \text{Im}\,\mathbb{H}$, define

$$T_q : \text{Im}\,\mathbb{H} \to \text{Im}\,\mathbb{H}, \quad T_q(v) = qvq^{-1}.$$

(i) Show that $T_q(v)$ is again a pure quaternion, that is, the map is well-defined.

(ii) Prove that T_q is $\mathbb{R}$-linear.

(iii) Show that $T_q \in SO(3)$.

$1 + 1 + 1 = 3$

Problem 4

With respect to the basis $\{e_1 = \mathbf{i}, e_2 = \mathbf{j}, e_3 = \mathbf{k}\}$, find out the the matrix of T_q, and hence get a homomorphism

$$\psi : S^3 \to SO(3), \quad q \mapsto [T_q].$$

(i) Show that the $\ker \psi = \{\pm 1\}$. (By $\pm 1 = (\pm 1, 0, 0, 0)$).

(ii) Also, show that ψ is surjective onto $SO(3)$. (Hint: any rotation in $\mathbb{R}^3$ is rotation about some axis by some angle).

(iii) Conclude that $S^3/\{\pm 1\} \cong SO(3)$ and hence conclude that $SU(2)/\pm I \cong SO(3)$.

$1.5 + 3 + 1.5 = 6$