MATRIX GROUPS

(MTH565)

Quiz 4: Solution

Thursday, 18th September 2025

Good Luck!

Problem Set

Problem 1

Prove or disprove:

(i) $O(5)$ is isomorphic to $SO(5) \times \{1, -1\}$.

(ii) $O(2)$ is isomorphic to $SO(2) \times \{1, -1\}$.

$\boxed{3 + 2 = 5}$

Solution

(i) TRUE Define a map

$$\phi : O(5) \to SO(5) \times \{1, -1\}, \quad A \mapsto ((\det A) \cdot A, \det A).$$

Note that the map is well defined as if $\det A = -1$, then $\det(-A) = (-1)^5 \det A = 1$ and hence $\det A \cdot A \in SO(5)$.

(a) The map ϕ is a homomorphism. For that, take $A, B \in O(5)$, then

$$\begin{aligned}
\phi(AB) &= ((\det(AB) \cdot AB, \det(AB))) \\
&= (\det A \cdot \det B \cdot AB, \det A \cdot \det B) \\
&= (\det A \cdot A, \det A) \cdot (\det B \cdot B, \det B) \\
&= \phi(A) \cdot \phi(B).
\end{aligned}$$

(b) ϕ is injective. Note that if $A \in \ker \phi$, then

$$\begin{aligned}
\phi(A) = (I, 1) &\implies (\det A \cdot A, \det A) = (I, 1) \\
&\implies \det A \cdot A = I \text{ and } \det A = 1 \\
&\implies A = I.
\end{aligned}$$

Thus, $\ker \phi \subseteq \{I\}$. Also, $I \in \ker \phi$ which implies $\ker \phi = \{I\}$. This proves that ϕ is injective.

(c) For surjective, note that for any $A \in SO(5)$, $-A \in O(5)$. Thus,

$$\begin{aligned}
\phi(A) &= ((\det A) \cdot A, \det A) = (A, 1) \\
\phi(-A) &= ((\det(-A)) \cdot (-A), \det(-A)) = (A, -1)
\end{aligned}$$

where the last equality followed because the size of the matrix is 5. Thus, the map is surjective.

(ii) FALSE Note that $SO(2)$ is abelian as it is nothing but the rotations of circle, and $\{1, -1\}$ is also abelian, so $SO(2) \times \{1, -1\}$ is an abelian group, whereas $O(2)$ is not an abelain group. For example, consider rotation followed by a reflection and reflection followed by a rotation.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Also you can look at the elements of order 2 in both groups.

Problem 2

Define the *Affine group* as

$$\mathrm{Aff}_n(\mathbb{F}) := \left\{ \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} : A \in GL_n(\mathbb{F}) \text{ and } \mathbf{v} \in \mathbb{F}^n \right\}.$$

Given any $X = \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}_n(\mathbb{F})$, we can identify it with a functions $f(\mathbf{x}) = A\mathbf{x} + \mathbf{v}$ from $\mathbb{F}^n$ to $\mathbb{F}^n$. Define a translated line

$$\ell_{\mathbf{v}_0} = \{\mathbf{v}_0 + \mathbf{v} : \mathbf{v} \in W\},$$

where $\mathbf{v}_0 \in \mathbb{F}^n$ and $W \subset \mathbb{F}^n$ is an 1-dimensional $\mathbb{F}$-subspace. Prove that f sends translated lines in $\mathbb{F}^n$ to translated lines in $\mathbb{F}^n$.

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Solution

We need to show that $f(\ell_{\mathbf{v}_0})$ is a (translated) line in $\mathbb{F}^n$. Let $f \in \mathrm{Aff}_n(\mathbb{F})$ and write

$$f_{\mathbf{v}}(\mathbf{x}) = A\mathbf{x} + \mathbf{v},$$

for some $A \in GL_n(\mathbb{F})$ and $\mathbf{v} \in \mathbb{F}^n$. Let W be a 1-dimensional subspace of $\mathbb{F}^n$. Note that for any $\mathbf{v}_0 + \mathbf{y} \in \ell_{\mathbf{v}_0}$ with $\mathbf{y} \in W$,

$$\begin{aligned} f_{\mathbf{v}}(\mathbf{v}_0 + \mathbf{y}) &= A(\mathbf{v}_0 + \mathbf{y}) + \mathbf{v} = A\mathbf{v}_0 + A\mathbf{y} + \mathbf{v} \\ &= (A\mathbf{v}_0 + \mathbf{v}) + A\mathbf{y}. \end{aligned}$$

Since W is a 1-dimensional subspace of $\mathbb{F}^n$, and $A \in GL_n(\mathbb{F})$, the subspace $\{A\mathbf{y} : y \in W\}$ is also a 1-dimensional subspace of $\mathbb{F}^n$. Hence, $f(\ell_{\mathbf{v}_0})$ is a line $\ell_{A\mathbf{v}_0+\mathbf{v}}$

Problem 3

Recall that the translational group is defined as

$$\mathrm{Trans}(\mathbb{R}^n) = \{f \in \mathrm{Isom}(\mathbb{R}^n) : f(\mathbf{x}) = \mathbf{x} + \mathbf{v}, \mathbf{v} \in \mathbb{R}^n\}.$$

(i) Show that $\mathrm{Trans}(\mathbb{R}^n)$ can be thought as a subset of $GL_{n+1}(\mathbb{R})$.

(ii) Assume that $\mathrm{Trans}(\mathbb{R}^n)$ is a subgroup of $\mathrm{Isom}(\mathbb{R}^n)$, show that $\mathrm{Trans}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Isom}(\mathbb{R}^n)$.

$1 + 3 = 4$

Solution

Given that

$$\text{Trans}(\mathbb{R}^n) = \{f_{\mathbf{v}} \in \text{Isom}(\mathbb{R}^n) : f(\mathbf{x}) = \mathbf{x} + \mathbf{v}, \mathbf{v} \in \mathbb{R}^n\}.$$

(i) Note that we can include $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, as $\mathbf{x} \mapsto \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$. So, the function f can be viewed as the following matrix:

$$f_{\mathbf{v}}(\mathbf{x}) = \begin{pmatrix} I_n & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix},$$

where I_n is the identity matrix of size n. Note that the determinant of the matrix $\begin{pmatrix} I_n & \mathbf{v} \\ 0 & 1 \end{pmatrix} = 1$ and hence this is an invertible matrix. Thus,

$$\text{Trans}(\mathbb{R}^n) = \left\{ \begin{pmatrix} I_n & \mathbf{v} \\ 0 & 1 \end{pmatrix} : \mathbf{v} \in \mathbb{R}^{n+1} \right\} \subset GL_{n+1}(\mathbb{R})$$

(ii) We need to show that $\text{Trans}(\mathbb{R}^n)$ is a normal subgroup of $\text{Isom}(\mathbb{R}^n)$. We take $f_{\mathbf{v}} \in \text{Trans}(\mathbb{R}^n)$ and $g \in \text{Isom}(\mathbb{R}^n)$ and we will show that $g f_{\mathbf{v}} g^{-1} \in \text{Trans}(\mathbb{R}^n)$. We have seen that any isometry of $\mathbb{R}^n$ can be viewed as

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{v}, \quad A \in O(n) \text{ and } \mathbf{v} \in \mathbb{R}^n$$

and the matrix representation is

$$\text{Isom}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} : A \in O(n), \mathbf{v} \in \mathbb{R}^n \right\}.$$

We need to show that for $A \in O(n)$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{w} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix}^{-1} \in \text{Trans}(\mathbb{R}^n).$$

Note that,

$$\begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\mathbf{v} \\ 0 & 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{w} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{w} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}\mathbf{v} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}\mathbf{v} + \mathbf{w} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I & A\mathbf{w} \\ 0 & 1 \end{pmatrix} \in \text{Trans}(\mathbb{R}^n). \end{aligned}$$