Matrix Groups

(MTH565)

Quiz 3: Solution

Monday, 4^{th} September 2025

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Good Luck!

Problem Set

Problem 1

True/False problems. If the statement is true, then prove it otherwise provide a counterexample or disprove it.

Let D_r be the set of $n \times n$ real matrices with determinant r.

(i) D_0 is a closed set in $M_n(\mathbb{R})$.

(ii) $GL_n(\mathbb{R})$ is a closed set in $M_n(\mathbb{R})$.

(iii) $\bigcup_{r\in\mathbb{R}\setminus\{0\}} D_r$ is compact in $M_n(\mathbb{R})$.

(iv) $O_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

(v) A continuous function maps a bounded set to bounded set.

$\boxed{1+2+1+2+1=6}$

Solution

We have seen in the class that the map

$$\det : M_n(\mathbb{R}) \to \mathbb{R}$$

is continuous (as it is a polynomial in n^2-variables).

(i) $\boxed{\text{TRUE}}$ The set D_0 can be seen as the inverse of 0 under the det map, that is,

$$D_0 = \det^{-1}(0).$$

Since $\{0\}$ is closed in $\mathbb{R}$ and det is a continuous function, so D_0 is closed in $M_n(\mathbb{R})$.

(ii) $\boxed{\text{FALSE}}$ We have seen that a set $K \subseteq X$ is closed if given any convergent seqeunce in K, the limit must belongs to K. We take a sequence of matrices in $GL_n(\mathbb{R})$ as

$$A_k = \frac{1}{k} \cdot I_n \in GL_n(\mathbb{R}).$$

But the limit

$$\lim_{k\to\infty} A_k = \begin{bmatrix} \lim\limits_{k\to\infty} \frac{1}{k} & 0 & \cdots & 0 \\ 0 & \lim\limits_{k\to\infty} \frac{1}{k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lim\limits_{k\to\infty} \frac{1}{k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \notin GL_n(\mathbb{R}).$$

(iii) $\boxed{\text{FALSE}}$ Note that

$$\bigcup_{r\in\mathbb{R}\setminus\{0\}} D_r = GL_n(\mathbb{R}).$$

To show this, take $A \in GL_n(\mathbb{R})$, then $A \in D_{\det A}$ which implies $GL_n(\mathbb{R}) \subseteq \cup D_r$. For the converse, take A in the union, which implies there exists $r \in \mathbb{R} \setminus \{0\}$ such that $A \in D_r$. Thus, $\det A = r \neq 0$ and hence $A \in GL_n(\mathbb{R})$.

Since, a compact set has to be closed, but $GL_n(\mathbb{R})$ is not closed, we conclude that it is not compact.

(iv) $\boxed{\text{TRUE}}$ Consider the map

$$\mathcal{T} : M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad A \mapsto A^T A.$$

In the homework problem, we have seen that $A \mapsto A^T$ and $(A, B) \mapsto A \cdot B$ are continuous functions. Thus, we can see the map $\mathcal{T}$ as a composition of

$$M_n(\mathbb{R}) \xrightarrow{f} M_n(\mathbb{R}) \times M_n(\mathbb{R}) \xrightarrow{g} M_n(\mathbb{R}), \quad f(A) = (A^T, A) \text{ and } g(A, B) = A \cdot B.$$

Since f and g are continuous and $\mathcal{T} = g \circ f$ is continuous. Observe that

$$O_n(\mathbb{R}) = \mathcal{T}^{-1}(I).$$

Thus, $O_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

(v) $\boxed{\text{FALSE}}$ Consider the function

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{1}{x}.$$

The function f is continuous but the image of $(0, 1) = (1, \infty)$ which is not bounded.

Problem 2

Consider the set of orthogonal matrices with real entries, that is, $O_n(\mathbb{R})$. We say that a set $X \subseteq O_n(\mathbb{R})$ is open (closed) in $O_n(\mathbb{R})$ if there exists an open (closed) set $K \subseteq M_n(\mathbb{R})$ such that $X = K \cap O_n(\mathbb{R})$.

(i) Is $SO(n)$ closed in $O_n(\mathbb{R})$?

(ii) Is it open in $O_n(\mathbb{R})$?

$\boxed{2 + 2 = 4}$

Solution

Similar to Problem 1 (part 1) we can show that D_1 and D_{-1} is closed in $M_n(\mathbb{R})$.

(i) Note that

$$SO(n) = D_1 \cap O_n(\mathbb{R}).$$

Thus, $SO(n)$ is closed in $O_n(\mathbb{R})$.

(ii) Similarly,

$$O_n(\mathbb{R}) - SO(n) = D_{-1} \cap O_n(\mathbb{R}),$$

which implies the complement of $SO(n)$ is $O_n(\mathbb{R})$ is closed and hence $SO(n)$ is open in $O_n(\mathbb{R})$.

Problem 3

Consider the dot product in $\mathbb{R}^n$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i, \text{ for } \mathbf{x} = (x_1, \ldots, x_n) \text{ and } \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

Prove that a matrix $A \in O_n(\mathbb{R})$ if and only if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.

$\boxed{2}$

Solution

We observe that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}.$$

Let $A \in O_n(\mathbb{R})$, then $AA^T = I = A^T A$. Now consider

$$\langle A\mathbf{x}, A\mathbf{y} \rangle = (A\mathbf{y})^T A\mathbf{x} = \mathbf{y}^T A^T A\mathbf{x} = \mathbf{y}^T \mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle.$$

For the converse note that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \implies \mathbf{y}^T A^T A\mathbf{x} = \mathbf{y}^T \mathbf{x} \implies \mathbf{y}^T (A^T A - I)\mathbf{x} = 0.$$

We take $\mathbf{x} = e_i$ and $\mathbf{y} = e_j$, then

$$\mathbf{y}^T (A^T A - I)\mathbf{x} = (A^T A - I)_{ij} = 0.$$

This implies all the entries of $A^T A - I$ is zero and hence $A^T A - I = 0$. Thus, $A \in O_n(\mathbb{R})$.