Matrix Groups

(MTH565)

Quiz 2: Solution

Monday, 28th August 2025

Good Luck!

Problem Set

Problem 1

(i) Determine the groups $GL_1(\mathbb{C}), SL_1(\mathbb{C}), O_1(\mathbb{C})$ and $SO_1(\mathbb{C})$.

$\boxed{0.5 \times 4 = 2}$

(ii) Find the inverse of the matrix $\begin{bmatrix} 5 & 3 \\ 2 & 3 \end{bmatrix}$ in $GL_2(\mathbb{Z}_{11})$.

$\boxed{2}$

Solution

(i) The groups are as follows:

- $GL_1(\mathbb{C}) = \{z \in \mathbb{C} : z \neq 0\} = \mathbb{C}^*$.
- $SL_1(\mathbb{C}) = \{z \in \mathbb{C} : z = 1\} = \{1\}$ = The trivial group.
- $O_1(\mathbb{C}) = \{z \in \mathbb{C} : [z] \cdot [z]^T = 1\} = \{z \in \mathbb{C} : z^2 = 1\} = \{\pm 1\} = \mathbb{Z}_2$.
- $SO_1(\mathbb{C}) = \{z \in O_1(\mathbb{C}) : \det[z] = 1\} = \{1\}$ = The trivial group

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(ii) Let the given matrix be A. Then A^{-1} in $GL_2(\mathbb{Z}_{11})$ would be

$$\begin{bmatrix} 5 & 3 \\ 2 & 3 \end{bmatrix}^{-1} = (\det A)^{-1} \begin{bmatrix} 3 & -3 \\ -2 & 5 \end{bmatrix} = 9^{-1} \begin{bmatrix} 3 & 8 \\ 9 & 5 \end{bmatrix}$$
$$= 5 \cdot \begin{bmatrix} 3 & 8 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 40 \\ 45 & 25 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 1 & 3 \end{bmatrix}.$$

Problem 2

(i) Prove that $SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}$.

$\boxed{2}$

(ii) Show that $SO(3)$ is a normal subgroup of $O(3)$. Further identify the quotient group $O(3)/SO(3)$ (you need to prove that which space this quotient group is isomorphic to).

$\boxed{1 + 2 = 3}$

Solution

(i) Recall that

$$SU(2) = \{A \in GL_2(\mathbb{C}) : AA^* = I_2 = A^*A \text{ and } \det A = 1\},$$

where A^* is the conjugate transpose of A. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

But, from the relation $AA^* = I_2$, we have $A^{-1} = A^*$. Thus,

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} \implies \bar{a} = d \text{ and } \bar{b} = -c.$$

Now, using the codition that the determinant is 1, we have $ad - bc = 1$. With the above conditions, we obtained

$$a\bar{a} - b(-\bar{b}) = 1 \implies |a|^2 + |b|^2 = 1.$$

Therefore, the set will be

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}$$

(ii) Let $A \in SO(3)$ and $B \in O(3)$. Recall that a subgroup $H \subseteq G$ is normal iff given any $h \in H$ and $g \in G$, $ghg^{-1} \in H$. We need to show that $BAB^{-1} \in SO(3)$. Since $\det A = 1$ and determinant is multiplicative,

$$\det(BAB^{-1}) = \det B \cdot \det A \cdot (\det B^{-1}) = \det B \cdot \det A \cdot (\det B)^{-1} = 1.$$

Since $O(3)$ is a group and $SO(3) \subseteq O(3)$, the matrix $BAB^{-1} \in O(3)$ and with the condition that $\det(BAB^{-1}) = 1$, $BAB^{-1} \in SO(3)$ and hence normal.
Consider a map

$$O(3) \xrightarrow{\varphi} \mathbb{Z}_2 = \{\pm 1\}, A \mapsto \det A.$$

Since $\det(I_3) = 1$ and $\det(-I_3) = -1$, the map is surjective homomorphism. The kernel is

$$\ker \varphi = \{A \in O(3) : \det A = 1\} = SO(3).$$

Thus, by first isomorphism theorem of groups, $O(3)/SO(3) \cong \mathbb{Z}_2$.

Problem 3

(i) Is $SL_2(\mathbb{Z})$ a subgroup of $GL_2(\mathbb{R})$ with usual matrix multiplication operation?

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(ii) Write all the elements of $O_2(\mathbb{Z})$ and $SO_2(\mathbb{Z})$.

$0.5 \times 2 = 1$

Solution

(i) We will prove that $SL_2(\mathbb{Z})$ is a subgroup of $GL_2(\mathbb{R})$. Since $\mathbb{Z} \subseteq \mathbb{R}$, it is clear that $SL_2(\mathbb{Z}) \subseteq GL_2(\mathbb{R})$. We only need to verify that the multiplication is closed and inverse exist. Since $\mathbb{Z}$ is closed under multiplication and addition, for any $A, B \in SL_2(\mathbb{Z})$, their product $AB \in GL_2(\mathbb{Z})$. Also, $\det(AB) = \det A \cdot \det B = 1$ and hence $AB \in SL_2(\mathbb{Z})$. To see the inverse, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. Then the inverse will be

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in SL_2(\mathbb{Z}).$$

Thus, $SL_2(\mathbb{Z})$ is a subgroup of $GL_2(\mathbb{Z})$.

(ii) In homework 4, Problem 4, we have seen that

$$O_2(\mathbb{R}) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}.$$

It will be in $O_2(\mathbb{Z})$ if each of the entries is an integer, which means $\cos\theta = 0, \pm 1$ and $\sin\theta = 0, \pm 1$. Thus, $\theta = \frac{n\pi}{2}, n \in \mathbb{Z}$. So,

$$O_2(\mathbb{Z}) = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix}, \begin{bmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\}.$$

If you want to see it geometrically, then it is as follows:

- Identity: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Rotation by 90°: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- Rotation by 180°: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Rotation by 270°: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
- Reflection about x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Reflection about y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Reflection about $y = x$ line: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- Reflection about $y = -x$ line: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Among the above matrices, all the rotation matrices have determinant 1 and hence

$$SO_2(\mathbb{Z}) = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \right\}.$$