MATRIX GROUPS

(MTH565)

Quiz 1: Solution

Wednesday, 20^{th} August 2025

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Good Luck!

Problem Set

Problem 1

Let us consider ordered bases of $\mathbb{R}^3$ as follows:

$$\mathcal{B} = \{(1,0,0),(3,3,3),(0,2,0)\}, \quad \mathcal{C} = \{(1,1,0),(0,0,1),(0,1,1)\}.$$

(i) Write the coordinates of vector $\mathbf{v} = (2,-1,3)$ with respect to basis $\mathcal{B}$.

(ii) Let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ denote the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Determine the matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

(iii) Hence, or otherwise, determine $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

(iv) What is the relationship between $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and $P_{\mathcal{C}\leftarrow\mathcal{B}}$?

(v) Calculate $P_{\mathcal{B}\leftarrow\mathcal{C}} \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$.

SOLUTION SPACE

Write your solution from the next page.

Begin Your Solution

(i) Writing the coordinates of $\mathbf{v}$ with respect to the basis $\mathcal{B}$:

$$\begin{aligned}\alpha(1,0,0)+\beta(3,3,3)+\gamma(0,2,0)=(2,-1,3) &\implies (\alpha+3\beta,3\beta+2\gamma,3\beta)=(2,-1,3)\\ &\implies \beta=1,\alpha=-1, \text{ and } \gamma=-2.\end{aligned}$$

Thus,

$$[\mathbf{v}]_{\mathcal{B}}=\begin{bmatrix}-1\\1\\-2\end{bmatrix}$$

(ii) In order to determine the change of basis matrix, we need to write the basis $\mathcal{B}$ in terms of the basis $\mathcal{C}$. That is, if

$$\mathcal{B}=\{\mathbf{v}_1,\dots,\mathbf{v}_n\} \text{ and } \mathcal{C}=\{\mathbf{w}_1,\dots,\mathbf{w}_n\},$$

then

$$P_{\mathcal{C}\leftarrow\mathcal{B}}=\Big[[\mathbf{v}_1]_{\mathcal{C}},\dots,[\mathbf{v}_n]_{\mathcal{C}}\Big].$$

Let

$$\mathbf{v}_1=(1,0,0), \mathbf{v}_2=(3,3,3), \text{ and } \mathbf{v}_3=(0,2,0).$$

For any $(a,b,c)\in\mathbb{R}^3$,

$$(a,b,c)=\alpha(1,1,0)+\beta(0,0,1)+\gamma(0,1,1)=(\alpha,\alpha+\gamma,\beta+\gamma).$$

Therefore,

$$\begin{aligned}(\alpha,\alpha+\gamma,\beta+\gamma)=(1,0,0) &\implies \alpha=1,\gamma=-1,\beta=1\\ (\alpha,\alpha+\gamma,\beta+\gamma)=(3,3,3) &\implies \alpha=3,\gamma=0,\beta=3\\ (\alpha,\alpha+\gamma,\beta+\gamma)=(0,2,0) &\implies \alpha=0,\gamma=2,\beta=-2.\end{aligned}$$

Hence,

$$P_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}1&3&0\\1&3&-2\\-1&0&2\end{bmatrix}$$

(iii) Similarly to determine $P_{\mathcal{B}\leftarrow\mathcal{C}}$, we will write each element of $\mathcal{C}$ in terms of $\mathcal{B}$. Again for any $(a,b,c)\in\mathbb{R}^3$, we have

$$(a,b,c)=\alpha(1,0,0)+\beta(3,3,3)+\gamma(0,2,0)=(\alpha+3\beta,3\beta+2\gamma,3\beta).$$

Therefore,

$$\begin{aligned}(\alpha+3\beta,3\beta+2\gamma,3\beta)=(1,1,0) &\implies \beta=0,\alpha=1,\gamma=\frac{1}{2}\\ (\alpha+3\beta,3\beta+2\gamma,3\beta)=(0,0,1) &\implies \beta=\frac{1}{3},\alpha=-1,\gamma=-\frac{1}{2}\\ (\alpha+3\beta,3\beta+2\gamma,3\beta)=(0,1,1) &\implies \beta=\frac{1}{3},\alpha=-1,\gamma=0.\end{aligned}$$

Thus,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

(iv) The matrices $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are inverses of each other, that is,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} P_{\mathcal{C}\leftarrow\mathcal{B}} = I_3.$$

(v) We want to calculate $P_{\mathcal{B}\leftarrow\mathcal{C}} \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$,

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$
