Matrix Group

Presentation Assignment

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Date: November 27, 2025
Duration: 30 minutes

Presentation Instructions

- The presentation contains 20 points. It is divided into three parts. The content contains 12 marks, whereas the presentation and the question answer contain 6 marks.
- The time limit is strict. You may take at most 5 minutes extra. So, in any case, try to wrap up your talk by 35 minutes.

Presentation Topic

The distance function on $O(n, \mathbb{R})$

The main aim is to understand the distance function $f : GL(n, \mathbb{R}) \to \mathbb{R}, \quad A \mapsto \text{dist}^2(A, O(n, \mathbb{R}))$.

Problem

Let $M(n, \mathbb{R})$ be the set of $n \times n$ real matrices, and $O(n, \mathbb{R})$ be the set of all orthogonal $n \times n$ matrices. Let $A, B \in M(n, \mathbb{R})$. We fix the standard Euclidean metric on $M(n, \mathbb{R})$ by identifying it with $\mathbb{R}^{n^2}$. This induces a distance function given by $\text{dist}(A, B) := \sqrt{\text{tr}(A-B)^T(A-B)}$. Consider the distance squared function

$$f : GL(n, \mathbb{R}) \to \mathbb{R}, \quad A \mapsto \operatorname{dist}^2(A, O(n, \mathbb{R})) = \inf_{B \in O(n,\mathbb{R})} \operatorname{dist}^2(A, B).$$

Show the following:

1. The function f can be explicitly expressed as

$$f(A) = n + \operatorname{tr}(A^T A) - 2 \operatorname{tr}\left(\sqrt{A^T A}\right).$$

2. The map $g : M(n, \mathbb{R}) \to \mathbb{R}, \quad A \mapsto \operatorname{tr}\left(\sqrt{A^T A}\right)$ is differentiable if and only if A is invertible.
3. Let A be a positive definite matrix and $\psi(A) = \sqrt{A}$. Then for any symmetric matrix H,

$$d\psi_A(H) = \int_0^\infty e^{-t\sqrt{A}} H e^{-t\sqrt{A}} \mathrm{d}t.$$

   Use this to show that for $A \in GL(n, \mathbb{R})$,

$$dg_A(H) = \left\langle A\left(\sqrt{A^T A}^{-1}\right), H\right\rangle$$

   for any symmetric matrix H.

Good luck with your presentation! If you have any questions, please don't hesitate to reach out.