Matrix Groups: Homework #13

Based on adjoint representation and covering map

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Problem 1

We have seen in the lecture that the map $\mathrm{Ad} : Sp(1) \to SO(3)$ is 2-to-1 map by looking at the kernel of the map. We have also proved that the map is a local diffeomorphism at I. In order to prove that it is a double covering map, we need to show that it is a local diffeomorphism at every point and it is surjective.

i) Show that Ad is a local diffeomorphism (use the left translation).

ii) Show that local diffeomorphism are open maps.

iii) Using (ii) and $SO(3)$ being connected, show that the map is surjective.

Problem 2

This problem describe the adjoint representation for $SO(3)$.

Consider a basis of $\mathfrak{so}(3)$ as

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

i) Show that

$$[E_1, E_2] = E_3, [E_2, E_3] = E_1, \text{and } [E_3, E_1] = E_2.$$

ii) Let $g = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. With the above basis, find the matrix of $\mathrm{Ad}_g : \mathfrak{so}_3 \to \mathfrak{so}_3$.

iii) Consider a vector space isomorphism

$$f : \mathbb{R}^3 \to \mathfrak{so}_3, \quad \boldsymbol{v} = (v_1, v_2, v_3) \mapsto v^\wedge := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$, Show that $\boldsymbol{v}^\wedge \boldsymbol{w} = \boldsymbol{v} \times \boldsymbol{w}$, where $\times$ is the cross product.

iv) For any $R \in SO(3)$, show that

$$R(\boldsymbol{v} \times \boldsymbol{w}) = R\boldsymbol{v} \times R(\boldsymbol{w}).$$

Hence or otherwise, conclude that

$$R\boldsymbol{v}^\wedge R^{-1} = (R\boldsymbol{v})^\wedge.$$

v) Express $(Re_i)^\wedge$ in the basis $\{E_1, E_2, E_3\}$ and conclude that Ad is an inclusion map.

Problem 3

i) Show that $\mathfrak{so}_3$ is not abelian and hence conclude that $SO(3)$ is not abelain.

ii) Show that $SO(3)$ is not abelain by finding two matrices in $SO(3)$ which do not commute.

iii) Use (ii), to prove that $SO(n)$ is not abelian for $n \geq 3$.

Problem 4

In this problem we will show that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$.

i) If G_1, G_2 are two matrix groups, show that $G_1 \times G_2$ is a matrix group. So, $Sp(1) \times Sp(1)$ is a matrix group.

ii) For any $\boldsymbol{v} \in \mathbb{H} \cong \mathbb{R}^4$, Consider the map

$$\varphi : Sp(1) \times Sp(1) \to GL_4(\mathbb{R}), \quad (g_1, g_2) \mapsto g_1 \boldsymbol{v} \bar{g}_2.$$

Show that the image will lie in $SO(4)$.

iii) By finding the kernel of the map, show that it is 2-to-1 map.

iv) Show that it is a local diffeomorphism (apply inverse function theorem)

v) Show that the map is surjective and hence conclude that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$.