Matrix Groups: Homework #12

Based on adjoint representation

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Theory

1. Recall that for any matrix group G, and for any $g \in G$, the conjugation
$$C_g : G \to G, \quad h \mapsto ghg^{-1}$$
is a smooth isomorphism and the derivative $\left(dC_g\right)_I : \mathfrak{g} \to \mathfrak{g}$ is denoted by Ad_g.
$$\mathrm{Ad}_g(X) = gXg^{-1}.$$
2. The *Lie bracket* of two vectors $A, B \in \mathfrak{g}$ is defined as
$$[A, B] := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{a(t)}(B) = AB - BA,$$
where $a(t)$ is any differentiable path in G with $a(0) = I$ and $a'(0) = A$.
3. Let $f : G_1 \to G_2$ be matrix group homomorphism with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$. Let $f : G_1 \to G_2$ be a smooth homomorphism. Then the derivative $\mathrm{d}f_I : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism.
4. The map Ad_g is a vector space isomorphism and hence induces a map
$$\mathrm{Ad} : G \to \mathrm{GL}(n, \mathbb{R}), \quad g \mapsto \mathrm{Ad}_g.$$
This is called the *adjoint representation* of G.
5. We can pass from the representation of the matrix group to its Lie algebra by taking the derivative at the identity, which we will denote by ad. For any $X \in \mathfrak{g}$,
$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [X, Y].$$

Problem 1

Use the definition of the Lie bracket to prove the ***Jacobi identity*** for a lie algebra $\mathfrak{g}$. That is, for any $A, B, C \in \mathfrak{g}$, show that
$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

Problem 2

i) Use (3) to show that smoothly isomorphic matrix groups have isomorphicc Lie algebras.

ii) We have seen that the converse need not be true. For example, $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ has the same Lie algebra but we will prove that the Lie groups are not isomorphic.

   a) Show that $SO(n, \mathbb{R})$ is a normal subgroup of $O(n, \mathbb{R})$ of index 2.

   b) $SO(n, \mathbb{R})$ does not have a normal subgroup of index 2.

Problem 3

The goal of this exercise is to show that for any $X \in \mathfrak{g}$, we have $\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}$.

i) Show that $(d\mathrm{Ad})_{I(X)} = \mathrm{ad}_X$ for any $X \in \mathfrak{g}$.

ii) Let G_1 and G_2 be two matrix groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Let $f : G_1 \to G_2$ be C^1 homomorphism. Prove that for all $v \in \mathfrak{g}_1$, $f(e^v) = e^{\mathrm{d}f_I(v)}$. Hence, conclude that $\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}$.

Problem 4

Let G_1, G_2 be matrix groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Suppose that $f : G_1 \to G_2$ is a smooth homomorphism. If $df_I : \mathfrak{g}_1 \to \mathfrak{g}_2$ is bijective, prove that $df_g : T_g G_1 \to T_{f(g)} G_2$ is bijective for all $g \in G_1$.

Problem 5

Let G be a path connected matrix group, and let U be a neighbourhood of I in G. Prove that U generates G, which means that every element of G is equal to a finite product $g_1 g_2 \dots g_k$ where $g_i \in U$ for $i = 1, 2, \dots, k$.

Problem 6

For a matrix group G of dimension n, prove that the function $\mathrm{Ad} : G \to GL(n, \mathbb{R})$ is smooth.