Matrix Groups: Homework #11

Based on manifolds

Dr. Sachchidanand Prasad

Problem 1

1. Let

$$\mathbb{S}^n = \left\{ (x_0, x_1, \ldots, x_n) : \sum_{i=0}^{n} x_i^2 = 1 \right\} \subset \mathbb{R}^{n+1}$$

be n-sphere. Let $N = (0, 0, \ldots, 1)$ and $S = (0, 0, \ldots, -1)$. Show that in the stereographic projection the maps are

$$\varphi : \mathbb{S}^n - \{N\} \to \mathbb{R}^n, \quad (x_0, x_1, \ldots, x_n) \mapsto \left(\frac{x_0}{1 - x_n}, \frac{x_1}{1 - x_n}, \ldots, \frac{x_{n-1}}{1 - x_n} \right)$$

and if $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, then

$$\varphi^{-1} : \mathbb{R}^n \to \mathbb{S}^n, \quad (y_1, y_2, \ldots, y_n) \mapsto \frac{2}{\|\boldsymbol{y}\|^2 + 1} (y_1, y_2, \ldots, \|\boldsymbol{y}\|^2 - 1).$$

Also, the maps from $\mathbb{S}^n - \{S\}$ is $\psi(\boldsymbol{x}) = -\psi(-\boldsymbol{x})$.

2. Show that the maps are smooth.

3. Compute the transition map $\psi \circ \varphi^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n - \{N\}, \varphi)$ and $(\mathbb{S}^n - \{S\}, \psi)$ defines a smooth structure on $\mathbb{S}^n$.

Problem 2

In this problem we will use the *regular value theorem* to prove that $\mathbb{S}^n$ is an n-dimensional manifold. Recall that for a smooth function $f : \mathbb{R}^m \to \mathbb{R}^n$, we say that $a \in \mathbb{R}^m$ is a *regular point* if df_a is of full rank (that is, the rank of df_a is n). The image $f(a)$ is called the *regular value*.

Consider the function

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \quad (x_0, x_1, \ldots, x_n) \mapsto \sum_{i=0}^{n+1} x_i^2.$$

1. Show that f is smooth.
2. Show that 1 is a regular value of f.
3. Hence, conclude that $\mathbb{S}^n$ is a smooth manifold of dimension n.

Problem 3

If M is a manifold of dimension n, then prove that any open subset $U \subset M$ is also an n-dimensional manifold.

Problem 4

Prove that the cone

$$C := \left\{ (x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2} \right\} \subset \mathbb{R}^3$$

is not a manifold.

Problem 5

Let $M_1 \subset \mathbb{R}^{d_1}$ and $M_2 \subset \mathbb{R}^{d_2}$ be two manifolds whose dimensions are m_1 and m_2, respectively. Prove that the product space

$$M_1 \times M_2 = \{(p_1, p_2) : p_1 \in M_1, p_2 \in M_2\} \subset \mathbb{R}^{d_1 + d_2}$$

is a manifold of dimension $m_1 + m_2$.