Matrix Groups: Homework #10

Based on multivariable calculus

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Problem 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) = \sqrt{|xy|}$. Show that f is not differentiable at $(0, 0)$.

Problem 2

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(\boldsymbol{x})| \le \|\boldsymbol{x}\|^2$. Show that f is differentiable at $\boldsymbol{x} = \boldsymbol{0}$.

Problem 3

Find f' for the following functions:

1. $f(x, y) = \sin(xy)$
2. $f(x, y, z) = (x^y, z)$
3. $f(x, y, z) = x^y$
4. $f(x, y, z) = x^{y^z}$
5. $f(x, y) = \int_a^{x+y} g(t)\mathrm{d}t$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Problem 4

Show (by an example) that the existence of all partial derivatives of a function does not imply differentiability of the function.

Problem 5

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ a continuously differentiable 1-1 function such that $\det df_{\boldsymbol{x}} \neq 0$ for all $\boldsymbol{x}$. Show that $f(U)$ is an open set and $f^{-1} : f(U) \to U$ is differentiable. Show also that f is an open map, that is, for any open set $V \subset \mathbb{R}^n$ $f(V)$ is open.

Problem 6

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f is not 1-1.

Generalize this result in the case of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ for $m < n$.

Problem 7

1. If $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'(a) \neq 0$ for all $a \in \mathbb{R}$, then show that f is 1-1.
2. Define

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) = (e^x \cos y, e^x \sin y).$$

Show that $\det df_{x,y} \neq 0$ for all $(x, y) \in \mathbb{R}^2$ but f is not 1-1.