Matrix Groups: Homework #9

Based on matrix exponential and Lie algebra

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Problem 1

For all $A, B \in M_n(\mathbb{K})$ with $A \in GL_n(\mathbb{K})$, show that

$$e^{ABA^{-1}} = Ae^B A^{-1}.$$

Problem 2

Prove that for any $A \in M_n(\mathbb{K})$,

$$\left(e^A\right)^* = e^{A^*}.$$

Problem 3

1. Let $A = \operatorname{diag}(a_1, a_2, ..., a_n) \in M_n(\mathbb{R})$. Calculate e^A. Using this, give a proof that $\det\left(e^A\right) = e^{\operatorname{trace}(A)}$.
2. A matrix A is said to be *similar* to B if there exists an invertible matrix P such that $B = P^{-1}AP$. Give a proof that $\det\left(e^A\right) = e^{\operatorname{trace}(A)}$ when A is similar to a diagonal matrix.

Problem 4

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Calculate e^A.

Problem 5

Find the Lie algebra of the following matrix groups.

1. $SO(n)$ and $\dim SO(n) = \frac{n(n-1)}{2}$.
2. $U(n)$ and $\dim U(n) = n^2$.
3. $Sp(n)$ and $\dim Sp(n) = 2n^2 + n$.

Problem 6

Recall that

$$UT_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K}) : A \text{ is upper triangular}\}.$$

Describe the Lie algebra of $UT_n(\mathbb{K})$.

Problem 7

Prove that the Lie algebra of $O(3)$ is isomorphic (as a vector space) to $\mathbb{R}^3$.

Problem 8

Let $G = SU(2)$. Denote

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $A_j = \iota \sigma_j$ for $j = 1, 2, 3$.

1. For $j = 1, 2, 3$, find $\exp(tA_j)$ and show that it is in G.
2. Explain why part (1) tell you that A_1, A_2 and A_3 are in the Lie algebra $\mathcal{L}(G)$.
3. Let

$$W = \left\{ \begin{pmatrix} u\iota & v + w\iota \\ -v + w\iota & -u\iota \end{pmatrix} : u, v, w \in \mathbb{R} \right\}$$

   show that every element of W can be written as a real linear combination of A_1, A_2 and A_3.
4. Explain why $\mathcal{L}(G) = W$.

Problem 9

Show that for $G = SU(n)$, the Lie algebra is

$$\mathcal{L}(G) = \mathfrak{su}(n) = \{A \in M(n, \mathbb{C}) : A^* + A = 0 \quad \text{and} \quad \operatorname{tr}(A) = 0\}.$$

Problem 10

For $\alpha = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $a \in \mathbb{R}$, show that $e^\alpha \in SO(2)$.