Matrix Groups: Homework #8

Based on orthogonal groups

Dr. Sachchidanand Prasad

Problem 1

We have seen that

$$O(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \bigcup \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

(1) Show that for every $A \in O(2) - SO(2)$, the corresponding linear map $R_A : \mathbb{R}^n \to \mathbb{R}^n$ is a flip about some line through the origin. How is this line determined by the angle of A.

(2) Let $B \in SO(2)$ and $\theta \notin \pi\mathbb{Z}$. Prove that B does not commute with any $A \in O(2) - SO(2)$.

(Hint: Show that R_{AB} and R_{BA} act differently on the line in $\mathbb{R}^2$ about which A is flipped.

Problem 2

Describe the product of two arbitrary elements of $O(2)$ in terms of their angles.

Problem 3

Let $A \in O(n)$ with determinant -1. Prove that

$$O(n) = SO(n) \cup \{A \cdot B : B \in SO(n)\}.$$

Problem 4

Define a map

$$f : O(n) \to SO(n) \times \{1, -1\}, \quad A \mapsto (\det(A) \cdot A, \det A).$$

(1) If n is odd, then f is an isomorphism.

(2) Prove that $O(2)$ is not isomorphic to $SO(2) \times \{1, -1\}$. Hint: Look for finite order elements.

Problem 5

Prove that $\mathrm{Tran}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Isom}(\mathbb{R}^n)$.