Matrix Groups: Homework #7

Based on orthogonal groups

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Problem 1

Let $f = f_n : \mathbb{C}^n \to \mathbb{R}^{2n}$ be defined as

$$f(a_1 + \iota b_1, ..., a_n + \iota b_n) = (a_1, b_1, ..., a_n, b_n).$$

(i) Show that for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}$,

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{\mathbb{C}} = \langle f(\boldsymbol{v}), f(\boldsymbol{w}) \rangle_{\mathbb{R}} + \iota \langle f(\boldsymbol{v}), f(\iota \boldsymbol{w}) \rangle_{\mathbb{R}}.$$

(ii) Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\} \subset \mathbb{C}$ is an orthogonal basis if and only if $\{f(\boldsymbol{v}_1), f(\iota \boldsymbol{v}_1), ..., f(\boldsymbol{v}_n), f(\iota \boldsymbol{v}_n)\}$ is an orthonormal basis of $\mathbb{R}^{2n}$.

Problem 2

In this problem we will prove the *Cauchy-Schwarz inequality*. For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}^n$,

$$|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| \leq \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\|.$$

Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}^n$. Let us denote $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =: \alpha$.

(i) Show that for any $\lambda \in \mathbb{F}$,

$$\|\lambda \boldsymbol{v} + \boldsymbol{w}\|^2 = |\lambda|^2 \|\boldsymbol{v}\|^2 + 2 \operatorname{Re}(\lambda \alpha) + \|\boldsymbol{w}\|^2.$$

(ii) Choose $\lambda = -\frac{\bar{\alpha}}{\|\boldsymbol{v}\|^2}$ to conclude

$$|\alpha| \leq \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\|.$$

(iii) Deduce when does the equality hold in the above inequality.

Problem 3

Remember the map $\rho_n : M_{n(\mathbb{C})} \to M_{2n}(\mathbb{R})$. Suppose instead of taking $f(a_1 + \iota b_1, ..., a_n + \iota b_n) = (a_1, b_1, ..., a_n, b_n)$ if we take $f(a_1 + \iota b_1, ..., a_n + \iota b_n) = (a_1, a_2, ..., a_n, b_1, ..., b_n)$ then how must ρ_n be defined so that the diagram commutes.