Matrix Groups: Homework #6

Based on topology of matrix groups

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Problem 1

Let (M, d) and (N, ρ) be two metric space and $f : M \to N$ be a conitnuous function. Prove or disprove the statements. If the statement is true, then prove it and if false, provide a counterexample.

1. If U is an open (closed) set in M, then $f(U)$ is open (closed) in N.
2. If K is boundedn in M, then $f(K)$ is also bounded in N.
3. If K is compactn in M, then $f(K)$ is also compact in N.
4. If K is bounded (compact) in N, then $f^{-1}(K)$ is bounded (compact) in M.

Problem 2

Let

$$M_r = \{A \in M_n(\mathbb{R}) : \det A = r\} \text{ and } T_r = \{A \in M_n(\mathbb{R}) : \operatorname{tr}(A) = r\},$$

where $\operatorname{tr} A$ denotes the trace of A.

1. Are M_r, T_r open? Are they closed?
2. Are M_r, T_r bounded?
3. Are M_r, T_r compact?

Problem 3

Let $M_n(\mathbb{R})$ is equipped with the Euclidean metric. Then prove that the following maps are conitnuous.

1. $\det : M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad A \mapsto \det A.$
2. $\operatorname{tr} : M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad A \mapsto \operatorname{tr} A.$
3. $\tau : M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad A \mapsto A^T.$
4. $\mu : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad (A, B) \mapsto A \cdot B.$
5. $\iota : GL_n(\mathbb{R}) \to GL_n(\mathbb{R}), \quad A \mapsto A^{-1}.$

Problem 4

We define the ***inner product*** on $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, as follows. Let $\boldsymbol{z} = (z_1, ..., z_n)$ and $\boldsymbol{w} = (w_1, ..., w_n)$. Then

$$\langle z, w \rangle := \sum_{j=1}^{n} z_j \bar{w}_j.$$

The ***norm*** on $\mathbb{F}^n$ is defined as $\|\boldsymbol{z}\| := \sqrt{\langle \boldsymbol{z}, \boldsymbol{z} \rangle}$. We define the following:

- Vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are called ***orthogonal*** if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

- The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are called ***orthonormal*** if they are orthogonal and $\|\boldsymbol{x}\| = 1 = \|\boldsymbol{y}\|$.
- A basis $\mathcal{B} = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ of a vector space V is called an ***orthonormal basis*** if $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$.

Now prove the following are equivalent.

(i) A matrix $A \in O_n(\mathbb{R})$, that is, $AA^T = I = A^T A$.

(ii) $\langle A\boldsymbol{x}, A\boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

(ii) The rows of A form an orthonormal basis of $\mathbb{R}^n$.