Matrix Groups: Homework #3

Based on groups

Dr. Sachchidanand Prasad

Problem 1

Let $M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices with real entries. Let $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{R}^n$. For $A \in M_n(\mathbb{R})$, define

$$R_A, L_A : \mathbb{R}^n \to \mathbb{R}^n, \quad R_A(\boldsymbol{x}) = \boldsymbol{x} \cdot A, \text{ and } L_A(\boldsymbol{x}) = \left(A \cdot \boldsymbol{x}^T\right)^T,$$

where $\boldsymbol{x}^T$ denotes the transpose of $\boldsymbol{x}$.

(i) Show that any linear function from $\mathbb{R}^n \to \mathbb{R}^n$ equals R_A for some $A \in M_n(\mathbb{R})$.

(ii) Show that any linear function from $\mathbb{R}^n \to \mathbb{R}^n$ equals L_A for some $A \in M_n(\mathbb{R})$.

(iii) We have seen $GL_n(\mathbb{R})$ is the set of all $n \times n$ invertible matrices over $\mathbb{R}$. Show that

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : R_A : \mathbb{R}^n \to \mathbb{R}^n \text{ is a linear isomorphism}\}.$$

(iv) Do you think in the previous part (iii), we can replace $\mathbb{R}$ with $\mathbb{H}$? That is, give any $A \in M_n(\mathbb{H})$, if det $A \neq 0$, then $R_A : \mathbb{H}^n \to \mathbb{H}^n$ is invertible.

(v) Find $A \in M_2(\mathbb{R})$ such that $R_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a counterclockwise rotation through an angle θ.

Problem 2

(i) Let $GL_2(\mathbb{Z})$ denote the set of all 2×2 matrices with integer entries and nonzero determinant. Is $GL_2(\mathbb{Z})$ a group with usual matrix multiplicaiton?

(ii) Let $SL_2(\mathbb{Z})$ denote the set of all 2×2 matrices with integer entries and determinant 1. Prove that $SL_2(\mathbb{Z})$ is a subgroup of $GL_2(\mathbb{R})$. Is it a normal subgroup?

(iii) Is $SL_n(\mathbb{Z})$ a subgroup of $GL_n(\mathbb{R})$?

Problem 3

Let G, H are groups. Let $\varphi : G \to H$ be a group homomorphism.

(i) Prove that ker φ is a normal subgroup of G.

(ii) Show that im φ is a subgroup of H. Is it a normal subgroup?

Problem 4

Describe a subgroup of $GL_{n+1}(\mathbb{R})$ that is isomorphic to the group $\mathbb{R}^n$ under the operation of vector addition.