Expansion of functions

Engineering Mathematics-I

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Today’s Goal

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- Taylor series and Maclaurin series

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- Taylor's polynomial

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$(x - 2)^{150}$	degree $= 150$	

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$e^x + \sin x - x^2$	there is no power of x

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and

$$a_i = b_i \quad \text{for } i = 0, 1, 2, \ldots, n.$$

Taylor's Expansion

If f has derivatives of all orders at $x = a$, then the *Taylor series* for the function f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

In the summation notation we can write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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$$n = 1 \quad f^{(1)}(0) = e^0 = 1 \quad \tfrac{1}{1!} x = x$$

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Thus,

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$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Maclaurin Series

In the Taylor's series, if we take $a = 0$, then the corresponding series is called *Maclaurin Series*.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

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2. Find the first four Taylor's polynomial for $f(x) = \frac{1}{x^2}$ at $x = 2$.
3. Find the first few terms of the Taylor series for the function $f(x) = \frac{1}{3}(2x + x\cos x)$ using power series operations.

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1. $f(x) = (1+x)^k, k \in \mathbb{N}$.

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Problem

Find the Maclaurin series for the function

$$f(x) = \sin\left(e^x - 1\right)$$

up to the term x^4.

More problems

- Using Taylor's theorem, express the polynomial

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  in powers of $(x - 1)$.
- Obtain the fourth-degree Taylor's polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \leq x \leq 0.5$.

Taylor's Theorem with Remainder

Write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots +$$

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where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some $c \in (a, x)$.

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> *The gamma function can be seen as a solution to the interpolation problem of finding a smooth curve $y = f(x)$ that connects the points of the factorial sequence $(x, y) = (n, n!)$ all positive integer values of n.*
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- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
- For any $n \geq 0$, $\Gamma(n+1) = n\Gamma(n)$.

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