Derivative

Engineering Mathematics-I

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SPNREC, Araria

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Today’s Goal

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- Lagrange's mean value theorem

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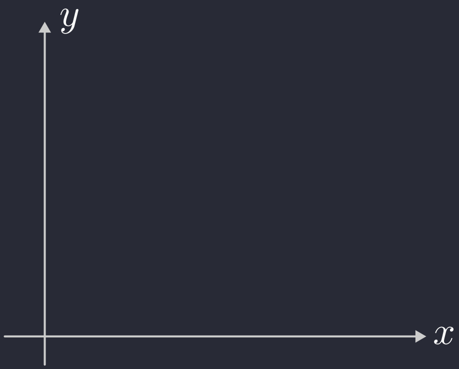

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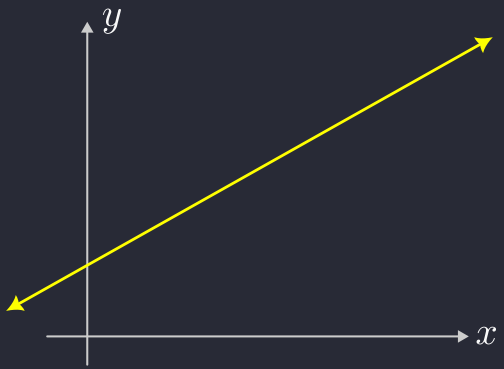

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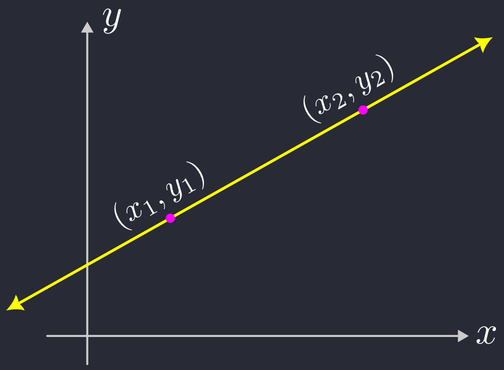

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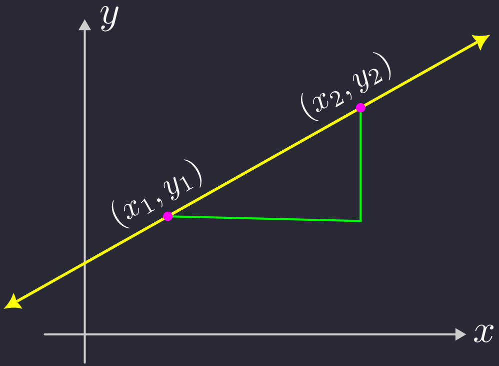

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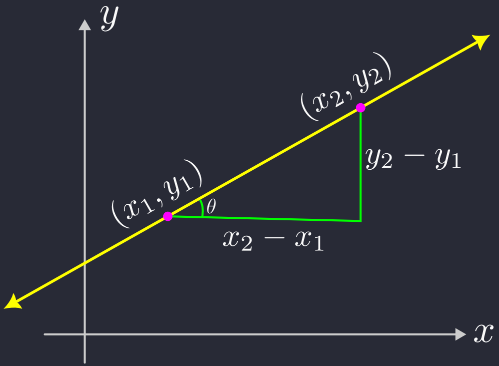

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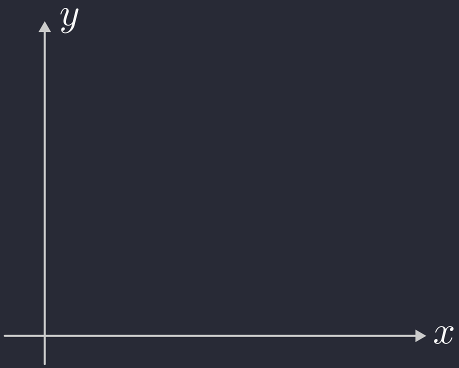

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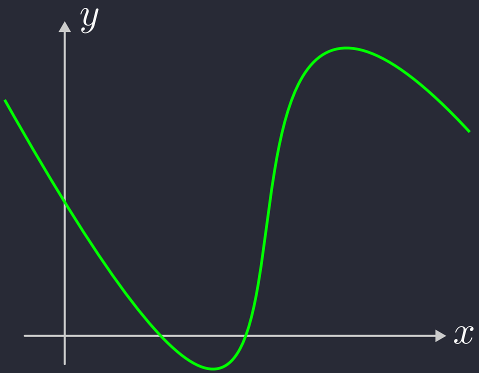

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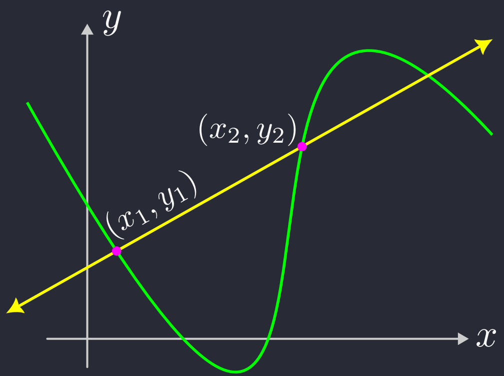

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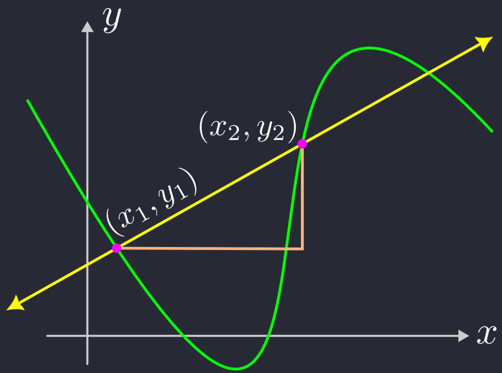

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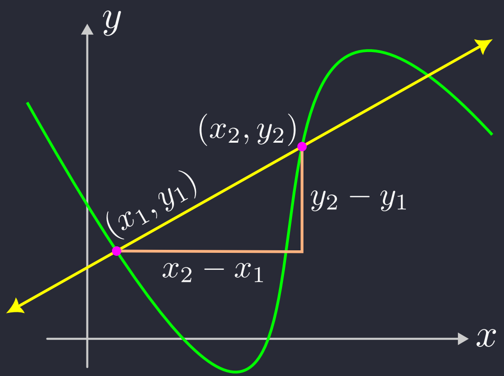

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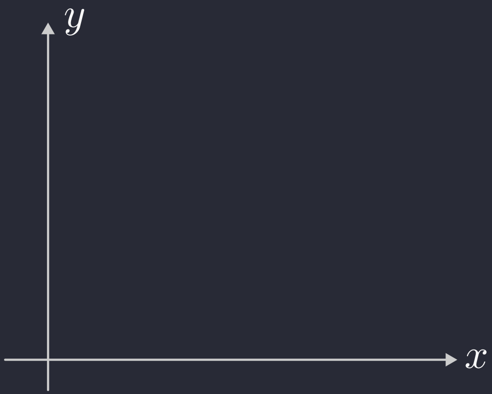

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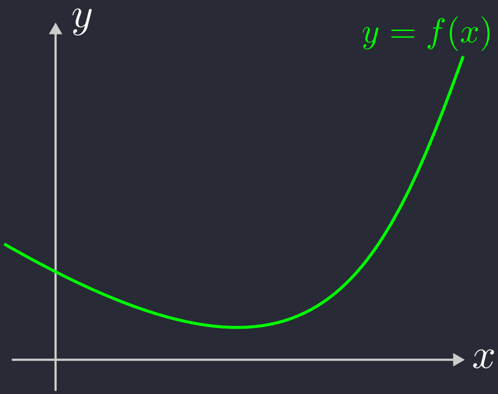

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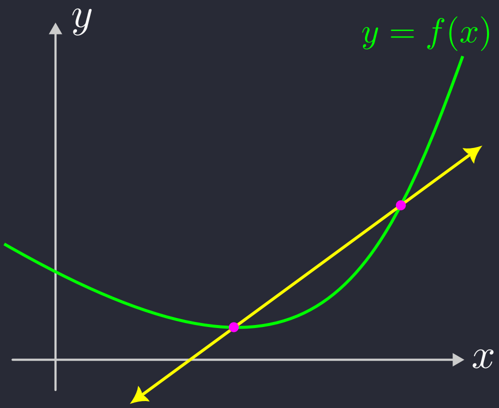

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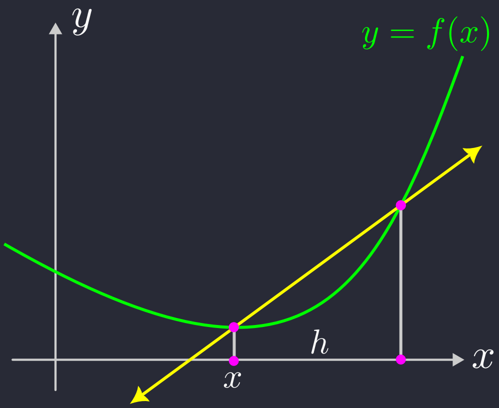

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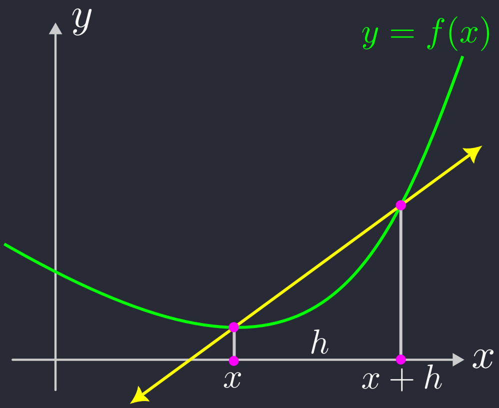

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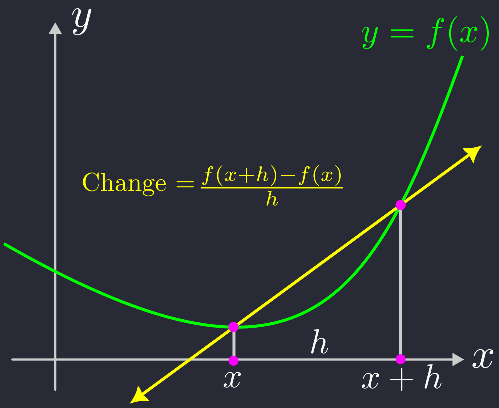

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- What will happen if h "tends" to 0?

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$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Slope of the tangent line is known as *Instantaneous Rate of Change*

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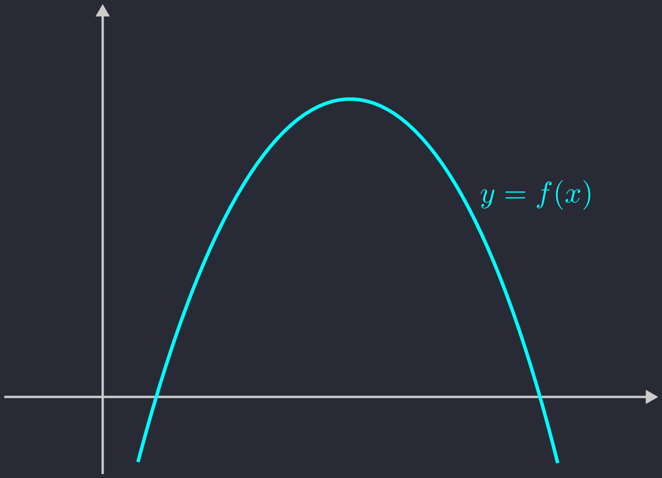

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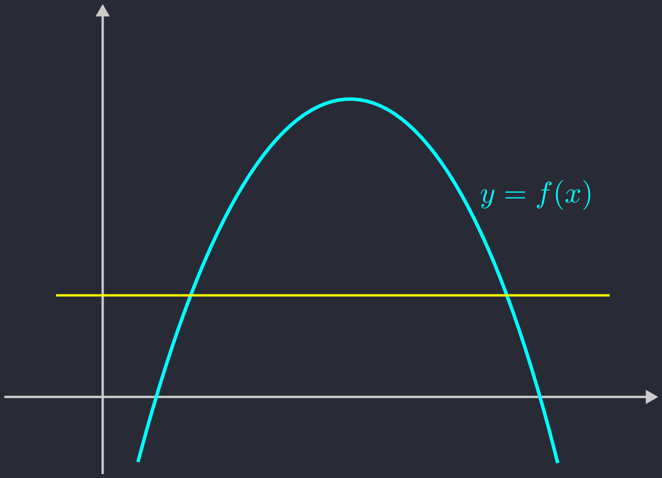

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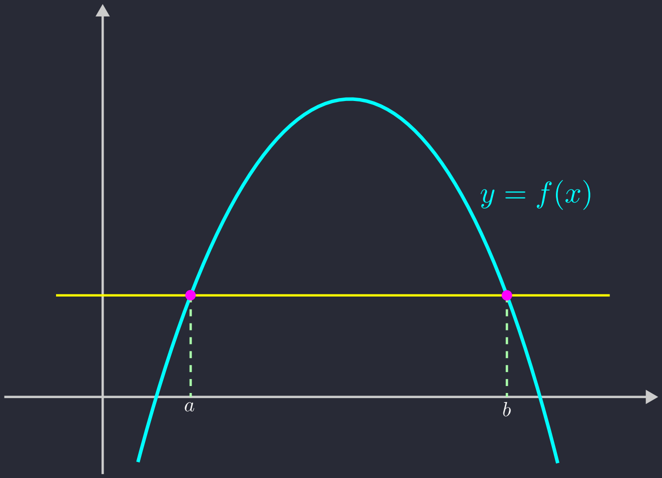

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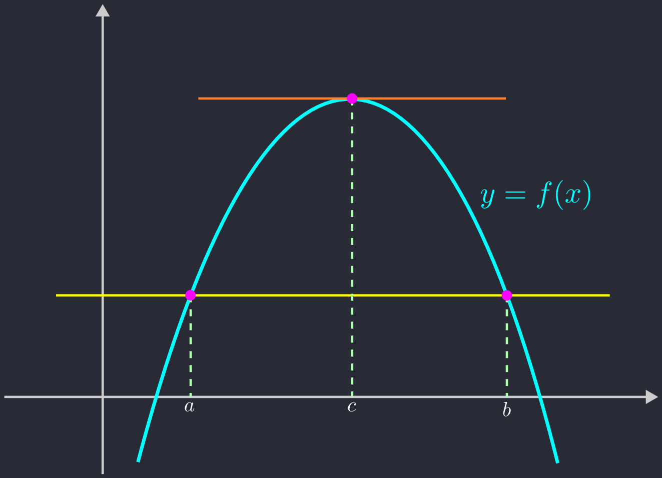

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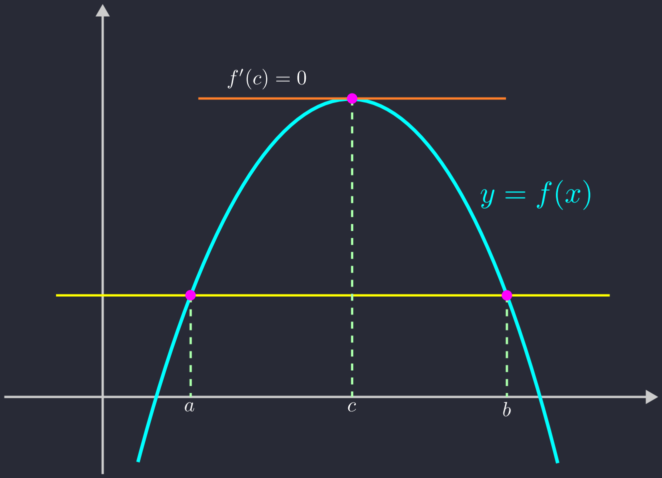

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Problems on Rolle's Theorem

Problem

For the function $f(x) = x\left(x^2 - 1\right)$ *test for the applicability of Rolle's theorem in the interval* $[-1, 1]$ *and hence find* c *such that* $-1 < c < 1$.

Problem cont...

Solution

Given that the function is

$$f(x) = x\left(x^2 - 1\right).$$

We have

1. *f is continuous on $[-1, 1]$,*

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3. $f(-1) = 0 = f(1)$.

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We have

1. f *is continuous on* $[-1, 1]$,
2. f *is differentiable on* $(-1, 1)$ *and*
3. $f(-1) = 0 = f(1)$.

Since f *satisfies the hypothesis of Rolle's theorem, there exists* $c \in (-1, 1)$ *such that* $f'(c) = 0$. *That is,*

Solution cont...

$$f'(c) = 0 \implies 3c^2 - 1 = 0$$

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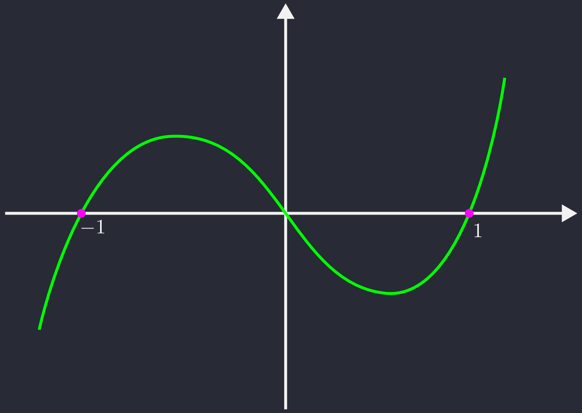

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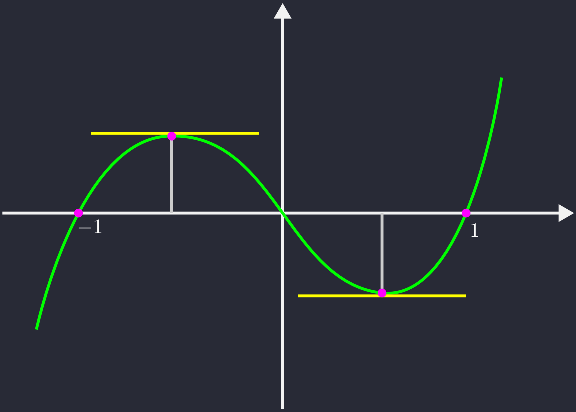

Problem 2

Problem

Verify the Rolle's theorem for

$$f(x) = \frac{\sin x}{e^x}, \quad in\ [0, \pi].$$

Solution

- Since $\sin x$ and e^x is continuous and e^x is not zero in the interval $[0, \pi]$, so $f(x) = \frac{\sin x}{e^x}$ is continuous on $[0, \pi]$.
- Since $\sin x$ and e^x is differentiable and e^x is not zero in the interval $(0, \pi)$, so $f(x) = \frac{\sin x}{e^x}$ is differentiable on $(0, \pi)$.
- Since

$$f(0) = \frac{\sin 0}{e^0} = 0 \quad \text{and } f(\pi) = \frac{\sin \pi}{e^\pi} = 0.$$

  So, Rolle's theorem is applicable for the given function and hence there exists $c \in (0, \pi)$ such that $f'(c) = 0$.

Problem 3

Problem

It is given that the Rolle's theorem holds for the function

$$f(x) = x^3 + bx^2 + cx, \quad 1 \leq x \leq 2$$

at the point $x = \frac{4}{3}$. *Find the value of* b *and* c.

Solution

- Since Rolle's theorem is applicable, so

$$\begin{aligned} f(1) = f(2) &\implies 1 + b + c = 8 + 4b + 2c \\ &\implies 3b + c = -7. \end{aligned}$$

  Also,

$$\begin{aligned} f'\left(\frac{4}{3}\right) = 0 &\implies \frac{16}{3} + \frac{8b}{3} + c = 0 \\ &\implies 8b + 3c = -16. \end{aligned}$$

- Solve the two equations to find b and c.

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Proof.
Take

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

□

A picture

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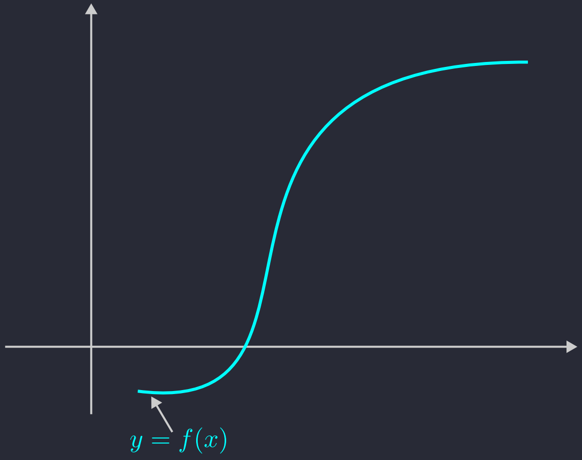

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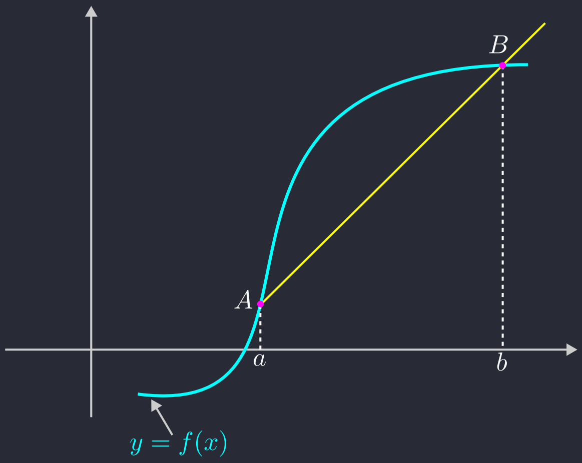

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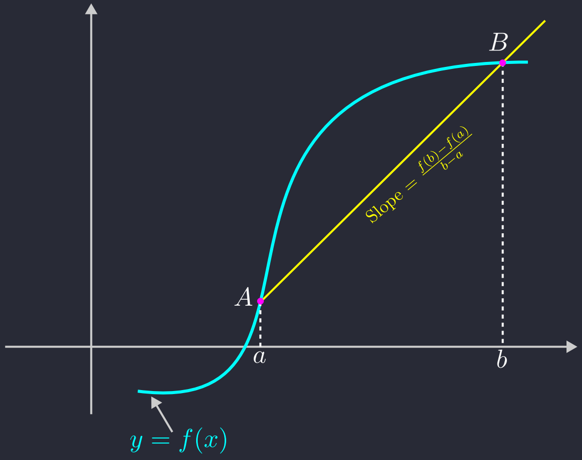
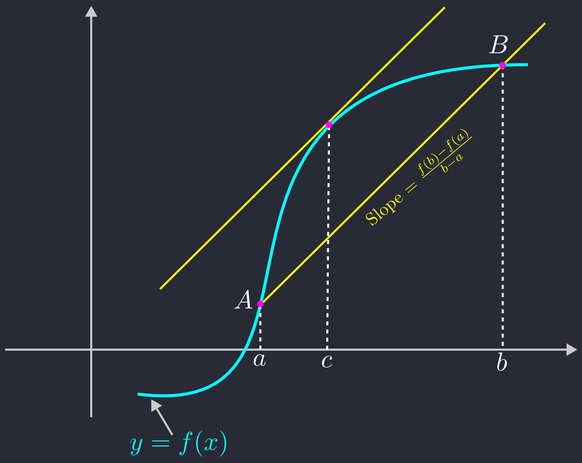
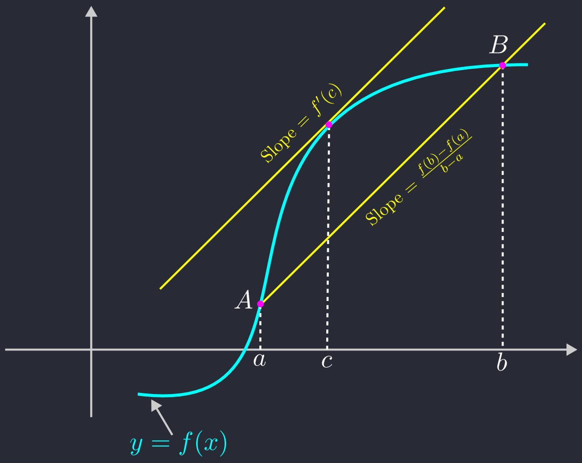


A picture



A picture



Problem-1

Problem

Verify the Lagrange's mean value theorem for the function

$$f(x) = x(x-1)(x-2), \quad a = 0 \text{ and } b = \frac{1}{2}.$$

Also find c.

Solution

- Since $f(x)$ is a polynomial it is continuous and differentiable.
- By the L.M.V.T there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Note that

$$f(x) = x^3 - 3x^2 + 2x \implies f'(x) = 3x^2 - 6x + 2.$$

Thus,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \implies 3c^2 - 6c + 2 = \frac{\frac{3}{8}}{\frac{1}{2}}$$

Solve for c.

Problem-2

Problem

Verify the Lagrange's mean value theorem for the function given below and find c

$$f(x) = \log x, \quad a = 1 \textit{ and } b = e.$$

Cauchy's Mean Value Theorem

Let f and g be two functions defined on $[a, b]$ such that

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- $g'(x) \neq 0$ for any $x \in (a, b)$.

Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$