Engineering Mathematics-1: Homework #3

Based on Taylor and Maclaurin Series

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Theory

This set of problems is based on Taylor and Maclaurin series. Let us recall both the series once (for details see in the lecture notes).

Taylor's Series

If f is an infinitely differentiable function (that is, we can differentiate infinitely many time) at some point a. Then the *Taylor series* of f at $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots.$$

Maclaurin Series

The *Maclaurin series* of f is the Taylor series at $x = 0$. That is,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots.$$

Problem 1

Find the first three Taylor polynomial of the function

$$f(x) = x^4 + x^2 + 1, \quad a = -2.$$

Problem 2

Find the Taylor polynomial of order 3 of the given funcitons at $x = a$.

1. $f(x) = e^{2x}, \quad a = 0$
2. $f(x) = \sin x, \quad a = 0$
3. $f(x) = \sin x, \quad a = \frac{\pi}{4}$
4. $f(x) = \tan x, \quad a = \frac{\pi}{4}$
5. $f(x) = \sqrt{1-x}, \quad a = 0$

Problem 3

Find the Maclaurin series of the following functions.

1. $f(x) = e^{-x}$
2. $f(x) = xe^{x}$
3. $f(x) = \frac{2+x}{1-x}$

4. $f(x) = \sin\left(\frac{x}{2}\right)$
5. $f(x) = x\cos x$

Theory

The next set of problems is based on the maximum error in a Taylor's polynomial.

Lagrange's Form of the Remainder

The remainder

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{(k+1)!}(x-a)^{n+1},$$

for some c between a and x is called *Lagrange's form of the remainder.*

Taylor's Inequality

If $\left|f^{(n+1)}(x)\right| \leq M$ for all x such that $|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in the interval $|x-a| \leq d$

Let us discuss an example, that we discussed in the class.

Problem: Obtain the fourth-degree Taylor's polynomial approximaiton to $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \leq x \leq 0.5$.

Solution: Let us first find the fourth degree Taylor's polynomial.

$$p_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4.$$

We have

$$\begin{aligned} f(x) &= e^{2x} \Rightarrow f(0) = 1 \\ f'(x) &= 2e^{2x} \Rightarrow f'(0) = 2 \\ f''(x) &= 4e^{2x} \Rightarrow f''(0) = 4 \\ f^{(3)}(x) &= 8e^{2x} \Rightarrow f^{(3)}(0) = 8 \\ f^{(4)}(x) &= 16e^{2x} \Rightarrow f^{(4)}(0) = 16. \end{aligned}$$

Thus, the required Taylor's polynomial will be

$$\begin{aligned} p(x) &= 1 + 2x + \frac{4}{2}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 \end{aligned}$$

We now find the maximum error. According to Taylor's theorem, there exists c between $a = 0$ and x such that

$$R_4(x) = \frac{f^{(5)}(c)}{5!}x^5.$$

Note that e^{2x} is an increaing funciton, So,

$$f^{(5)}(x) = 32e^{2x} \Rightarrow \max_{c \in (0,x) \text{ and } x \in (0,0.5)} 32e^{2c} = 32e^{2\times 0.5} = 32e.$$

So, using the Taylor's theorem,

$$|R_4(x)| \le 32e \max_{x \in (0,0.5)} \frac{x^5}{5!} = 32e\frac{0.5^5}{5!} = 0.00026e = 0.0007.$$

Problem 4

Calculate the second order Taylor polynomial for $f(x) = \sqrt{1+x}$ about $x = 0$, and write down a formula for the Remainder term $R_2(x)$. Hence show that

$$1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 \le \sqrt{1 - x^4} \le 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} \quad \text{for all } x \in \mathbb{R}.$$

Problem 5

Find the Maclaurin's series expansion of $f(x) = \tan^{-1} x$ up to four terms with Lagrange's form of the remainder.

Problem 6

Find the Taylor's series expansion of $\sin^2 x$ up to five terms with Lagrange's form of remainder.

Problem 7

Use Taylor's theorem to approximate value of $a = \sqrt{1.5}$ and $b = \cos 31^\circ$.

Problem 8

Use Taylor's theorem to prove that

$$x - \frac{x^2}{2} < \log(1+x) < x \qquad \text{for } x > 0.$$

Problem 9

Let $f(x) = \ln(1 + x)$. (Here $\ln x = \log_e x$)

1. Calculate the fourth order Taylor polynomial $T_4(x)$ for $f(x)$ centered at $x = 0$.
2. Use Taylor's Theorem to write down a formula for the fourth remainder term $R_4(x)$, and deduce that

$$\frac{x^5}{5(1+x)^5} \leq f(x) - T_4(x) \leq \frac{x^5}{5} \quad \text{for all } x \geq 0.$$

Problem 10

Express the polynomial

$$p(x) = x^4 + 3x^2 - 2x + 1$$

in powers of $(x - 2)$.