Complex Variables: Homework #5

Based on harmonic functions

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Important Tips

Harmonic Functions

We recall that a real valued function $f(x, y)$ is said to be *harmonic* in a domain D if

$$\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} = 0 \quad \left(\text{also written as } f_{xx} + f_{yy} = 0\right)$$

in D.

For example,

$$\begin{aligned} f(x, y) = x^2 - y^2 &\Rightarrow f_x = 2x \quad \text{and} \quad f_y = -2y \\ &\Rightarrow f_{xx} = 2 \quad \text{and} \quad f_{yy} = -2. \end{aligned}$$

Thus,

$$f_{xx} + f_{yy} = 2 - 2 = 0.$$

Thus, $f(x, y)$ is harmonic function.

Theorem

Suppose that the complex function $f(z) = u(x, y) = \iota v(x, y)$ is analytic in a domain D. Then the funcitons $u(x, y)$ and $v(x, y)$ are harmonic in D.

For example, $f(z) = z^2$, then

$$f(z) = z^2 = (x^2 - y^2) + \iota(2xy).$$

Thus,

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Then it is clear that

$$u_{xx} + u_{yy} = 0 \quad \text{nd} \quad v_{xx} + v_{yy} = 0.$$

Thus, real and imaginary part of the analytic function is harmonic.

Harmonic Conjugate Funcitons

Theorem

If $u(x, y)$ is a function which is harmonic in a domian D, then there exists a harmonic function $v(x, y)$ in D such that the function $f(z) = u(x, y) + \iota v(x, y)$ is analytic on D.

In the above, the function $v(x, y)$ is said to be *harmonic conjugate* of $u(x, y)$.

Theory

Let us discuss how to find an harmonic conjugate.

In the above we proved that the function $u(x, y) = x^2 - y^2$ is harmonic. Let us find its harmonic conjugate. That is, we need to find a harmonic function $v(x, y)$ such that $f(z) = u(x, y) + \iota v(x, y)$ is analytic. If $f(z) = u(x, y) = \iota v(x, y)$ is analytic, it must satisfies the Cauchy-Riemann equaitons. That is,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Consider the first equation, we get

$$\begin{aligned} u_x = v_y &\Rightarrow 2x = v_y \\ &\Rightarrow \int \frac{\partial v}{\partial y} dy = \int 2x dy \\ &\Rightarrow v(x, y) = 2xy + c(x), \end{aligned}$$

where $c(x)$ is a function of x only. Note that when we do integration we get the constant of the integration. Since, v is a function of x and y both, when we do integration with dy, the constant maybe a function of x also. We now will use the second equaiton.

$$\begin{aligned} u_y = -v_x &\Rightarrow -2y = -(2y + c'(x)) \\ &\Rightarrow -2y = -2y - c'(x) \\ &\Rightarrow c'(x) = 0 \\ &\Rightarrow c(x) = k, \quad \text{where } k \text{ is a constant.} \end{aligned}$$

Thus, the harmonic conjugate of u will be

$$v(x, y) = 2xy + k, \quad \text{where } k \text{ is a constant.}$$

The corresponding analytic function will be

$$f(z) = u(x, y) + \iota v(x, y) = x^2 - y^2 = \iota(2xy + k).$$

Problem 1

In the following problem verify that the given function is harmonic. If your answer is yes, then find the harmonic conjugate of u. Form the corresponding analytic function $f(z) = u + \iota v$.

1. $u(x, y) = x$
2. $u(x, y) = 2x - 2xy$
3. $u(x, y) = x^2 - y^2$
4. $u(x, y) = x^3 - 3xy^2$
5. $u(x, y) = \log_{e(x^2+y^2)}$
6. $u(x, y) = \cos x \cosh y$ (The derivative of $\cosh x$ is $\sinh x$)

7. $u(x, y) = e^x(x \cos y - y \sin y)$
8. $u(x, y) = -e^{-x} \sin y$

Problem 2

Find an analytic function $f(z)$ such that

$$\text{Re}(f'(z)) = 3x^2 - 4y - 3y^2 \quad \text{and} \quad f(1 + \iota) = 0.$$