Complex Variables: Homework #4

Based on analytic functions

Dr. Sachchidanand Prasad

Important Tips

We recall that if a function $f(z) = u(x, y) + \iota v(x, y)$ is analytic at $z_0 = x_0 + \iota y_0$, then it must satisfy the Cauchy-Riemann equaitons at z_0.

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

We can also write the Cauchy-Riemann equations in the polar form. For example, consider the function

$$f(z) = z^2 + 3z.$$

If we replace $z = x + \iota y$, we get

$$f(z) = (x^2 - y^2 + 3x) + \iota(2xy + 3y).$$

Thus,

$$u(x, y) = x^2 - y^2 + 3x \quad \text{and} \quad v(x, y) = 2xy + 3y.$$

Similarly, if we take $z = r(\cos\theta + \iota \sin\theta)$, then can write the function we can write $f(z) = u(r, \theta) + \iota v(r, \theta)$.

$$\begin{aligned} f(z) = z^2 + 3z &= (r(\cos\theta + \iota\sin\theta))^2 + 3(r(\cos\theta + \iota\sin\theta)) \\ &= r(\cos^2\theta - \sin^2\theta + 2\iota\cos\theta\sin\theta) + 3r\cos\theta + \iota 3r\sin\theta \\ &= r(\cos 2\theta + \iota\sin 2\theta) + 3r\cos\theta + \iota 3r\sin\theta \\ &= r(\cos 2\theta + 3r\cos\theta) + \iota(r(\sin 2\theta + 3\sin\theta)). \end{aligned}$$

Thus,

$$u(r, \theta) = r(\cos 2\theta + 3r\cos\theta) \quad \text{and} \quad v(r, \theta) = r(\sin 2\theta + 3\sin\theta).$$

The Cauchy-Riemann equations in the polar form is given by

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = -\frac{1}{r} u_\theta.$$

Theory

We recall some of the important complex functions.

1. Exponential funciton

$$e^z = e^{x+\iota\theta} = e^x(\cos\theta + \iota\sin\theta)$$

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2. Trigonometric functions

$$\cos z = \frac{e^{\iota z} + e^{-\iota z}}{2}, \quad \sin z = \frac{e^{\iota z} - e^{-\iota z}}{2\iota}$$

Problem 1

Check whether the following functions satisfy the Cauchy-Riemann equations.

1. $f(z) = \iota z\overline{z}$
2. $f(z) = e^{-2x}(\cos 2y - \iota \sin 2y)$
3. $f(z) = e^{x}(\cos y - \iota \sin y)$
4. $f(z) = \text{Re}(z^2) - \iota\ \text{Im}(z^2)$
5. $f(z) = \frac{1}{z - z^5}$
6. $f(z) = \frac{\iota}{z^8}$
7. $f(z) = \frac{3\pi^2}{z^3 + 4\pi^2 z}$.

To make easier, use the polar form of the Cauchy-Riemann equations in the last three problems.

Problem 2

Find the value of c_1 and c_2 such that the function

$$f(z) = x^2 + c_1 y^2 - 2xy + \iota(c_2 x^2 - y^2 + 2xy)$$

is analytic.

Problem 3

Find k such that the function $f(z)$ expressed in polar coordinates as

$$f(z) = r^2 \cos 2\theta + \iota r^2 k\theta$$

is analytic.