GATE 2025: Solution to Homework #1

Based on Functions

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Problem 1

Find the domain and range of each functions.

1. $f(x) = 1 + x^2$
2. $g(t) = \frac{2}{t^2-16}$
3. $h(s) = \sqrt{s^2 - 3s}$
4. $p(x) = \frac{4}{3-x}$
5. $s(x) = \sqrt{x^2+1}$

Solution

1. The given function is $f(x) = 1 + x^2$. Since the function is defined for all $x \in \mathbb{R}$, the domain of the function is $\mathbb{R}$.

   Let us see what will be range of this function. For any $x \in \mathbb{R}$,

   $$0 \leq x^2 < \infty \Rightarrow 1 \leq 1 + x^2 < \infty.$$

   Thus, the range of the fucntion will be $[1, \infty)$. This range can be also be found as follows. Let $y \in \mathbb{R}$ and y is in the range of f. Then there exists $x \in \mathbb{R}$ such that

   $$\begin{aligned} f(x) = y \Rightarrow 1 + x^2 &= y \\ \Rightarrow x^2 &= y - 1 \\ \Rightarrow x &= \pm\sqrt{y-1}. \end{aligned}$$

   The above expression is well defined if $y - 1 \geq 0$ which implies $y \geq 1$. Thus, the range will be $[1, \infty)$.

2. The given function is $g(t) = \frac{2}{t^2-16}$. This function will be well-defined if the denominator is nonzero. So, we must have

   $$t^2 - 16 \neq 0 \Rightarrow (t-4)(t+4) \neq 0 \Rightarrow t \neq \pm 4.$$

   Thus, the domian of the given function will be

   $$\text{Domain} = \mathbb{R} - \{\pm 4\} = (-\infty, -4) \cup (-4, 4) \cup (4, \infty).$$

   Now we will find the range of the function. If y is in the range of g, then there exists $t \in D(g)$ ($D(g) =$ domain of g) such that

   $$\begin{aligned} g(t) = y \Rightarrow \frac{2}{t^2-16} = y \qquad &\Rightarrow 2 = t^2 y - 16y \\ \Rightarrow t^2 y = 2 + 16y \quad &\Rightarrow t^2 = \frac{2+16y}{y} \\ \Rightarrow t = \pm\sqrt{\frac{2+16y}{y}}. \end{aligned}$$

The above expression is well defined if

$$\begin{aligned}
&\frac{2+16y}{y} \geq 0 \quad \text{and} \quad y \neq 0 \\
\Rightarrow &\begin{cases} 2+16y \geq 0 \text{ if } y > 0 \\ 2+16y \leq 0 \text{ if } y < 0 \end{cases} \quad \text{and} \quad y \neq 0 \\
\Rightarrow &\begin{cases} y \geq -\frac{1}{8} \text{ if } y > 0 \\ y \leq -\frac{1}{8} \text{ if } y < 0 \end{cases} \quad \text{and} \quad y \neq 0 \\
\Rightarrow &\begin{cases} y > 0 \\ y \leq -\frac{1}{8}. \end{cases}
\end{aligned}$$

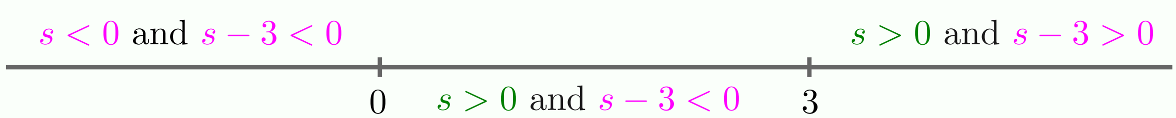
Thus, the range of the given function will be

$$R(g) = \left(-\infty, -\frac{1}{8}\right] \cup (0, \infty).$$

3. Te given funciton is $\sqrt{s^2 - 3s}$. For the domian of the function, we need

$$s^2 - 3s \geq 0 \Rightarrow s(s-3) \geq 0.$$

This is a product of two numbers, namely s and $s-3$. We break our anyalsis in three intervals shown below.



In the first and third intervals the sign of $s(s-3)$ is positve whereas in the second interval the sign is negative. Thus, the domian will be

$$D(h) = (-\infty, 0] \cup [3, \infty).$$

To find the range, we first note that on the domain $s^2 - 3s \geq 0$. Also, as s approaches to infinity, $s^2 - 3s$ also approcahes to infinity. Thus,

$$0 \leq s^2 - 3s < \infty \Rightarrow 0 \leq \sqrt{s^2 - 3s} < \infty.$$

Thus, the range of the function will be $[0, \infty)$. Note that we can also solve this problem similar to the earlier problems.

4. The given function is $p(x) = \frac{4}{3-x}$. The function is defined everywhere except when $3 - x = 0$. So, the domain of the function is $\mathbb{R} - \{3\}$. For the range, we observe that if $y \in \mathbb{R}$ such that

$$y = \frac{4}{3-x} \Rightarrow 3y - xy = 4 \Rightarrow x = \frac{3y-4}{y},$$

which is defined except at $y = 0$. Thus, the range will be

$$R(p) = \mathbb{R} - \{0\}.$$

5. The given function is $s(x) = \sqrt{x^2+1}$. Since for any $x \in \mathbb{R}$, the value of $x^2 + 1 > 0$. Thus, the domain of the given function will be $\mathbb{R}$. For the range, we observe that

$$x^2 + 1 \geq 1 \Rightarrow \sqrt{x^2+1} \geq 1.$$

Thus, the range will be $[1, \infty)$. This can also be solved similar to the earlier problems. Let $y \in \mathbb{R}$ be in the range. It is clear that $y \geq 1$. So, there exists $x \in \mathbb{R}$ such that

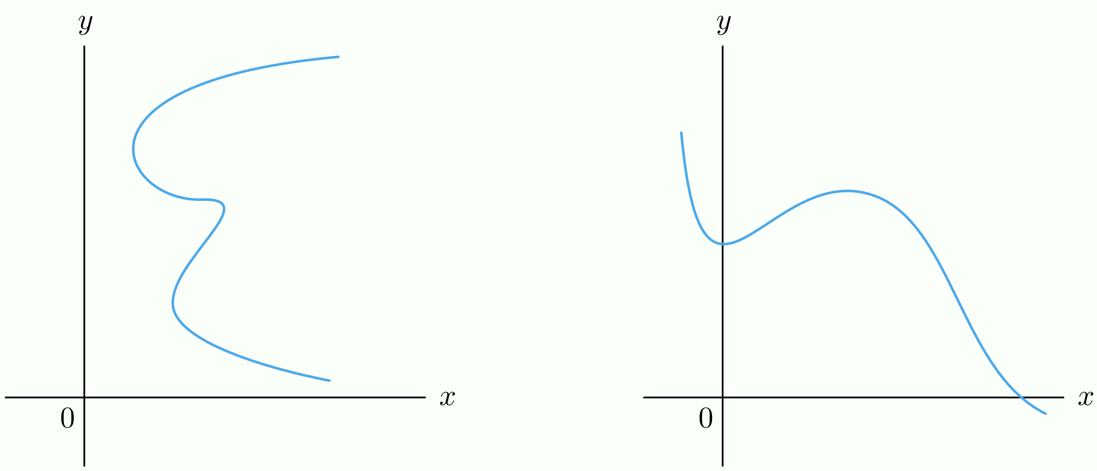
$$s(x) = y \Rightarrow \sqrt{x^2+1} = y \Rightarrow x^2 + 1 = y^2$$
$$\Rightarrow x = \pm\sqrt{y^2-1}.$$

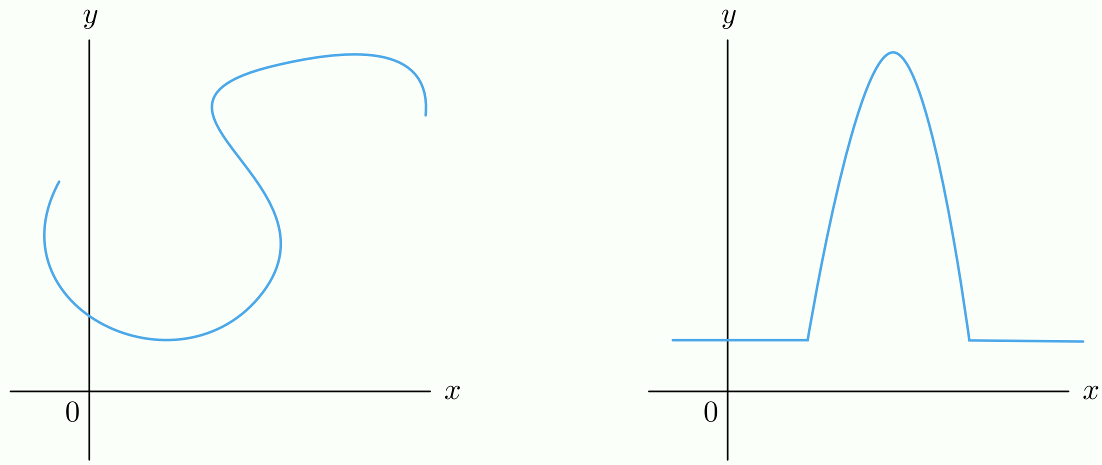
The above expression is well defined if $y^2 - 1 \geq 0$ which implies $(y-1)(y+1) \geq 0$. Similar to the third part of this problem, we will get $y \in (-\infty, -1] \cup [1, \infty)$. Since $y \geq 1$, the range will be

$$R(s) = [1, \infty).$$

Problem 2

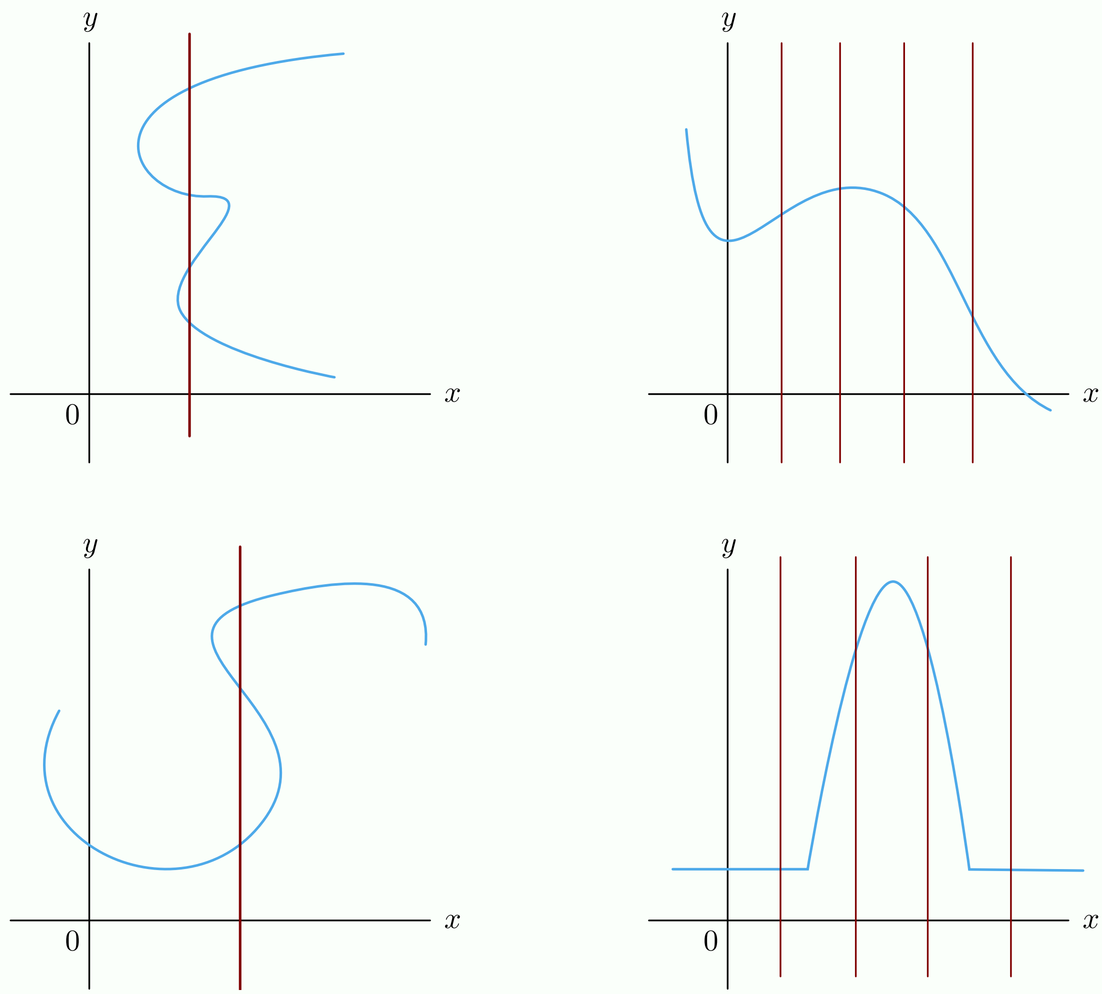
Which of the following are graphs of functions of x?





Solution

We will use vertical line test to check whether the given graph is a graph of some function of x.



It is clear that the first and third one can not be a graph of a function of x as the shown vertical line intersects the graph more than once. The other two graphs are the graphs of some function of x.

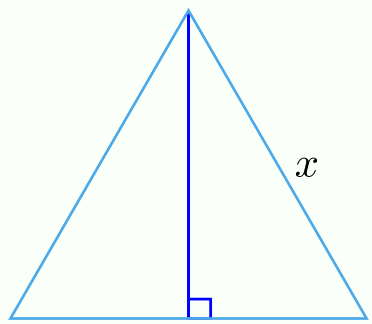
Problem 3

Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.

Solution

Since the side length of the equilateral triangle is x, the perimeter function will be

$$P : (0, \infty) \to (0, \infty), \quad p(x) = 3x.$$



Similarly, the area function will be

$$A : (0, \infty) \to (0, \infty), \quad A(x) = \frac{\sqrt{3}}{4}x^2.$$

Problem 4

Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let ℓ be the distance from the point (x, y) to the origin $(0, 0)$. Write ℓ as a function of x.

Solution

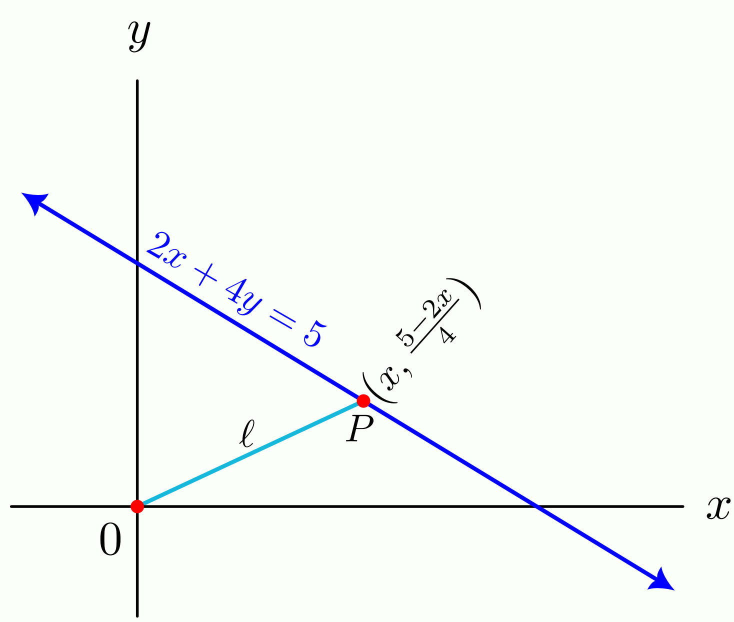
Take any point (x, y) on the line $2x + 4y = 5$. So, we can write

$$y = \frac{5 - 2x}{4}.$$

Thus, the point will be $\left(x, \frac{5-2x}{4}\right)$. The distance to this point to the origin will be

$$
\begin{aligned}
l(x) &= \sqrt{(x-0)^2 + \left(\frac{5-2x}{4} - 0\right)^2} \\
&= \sqrt{x^2 + \left(\frac{5-2x}{4}\right)^2} \\
&= \sqrt{x^2 + \frac{25-20x+4x^2}{16}} \\
&= \sqrt{\frac{16x^2+25-20x+4x^2}{16}} \\
&= \frac{\sqrt{(20x^2-20x+25)}}{4}
\end{aligned}
$$

Look at the figure below.



Problem 5

Find the domain of each functions.

1. $f(x) = \frac{x+3}{4-\sqrt{x^2-9}}$.
2. $g(t) = \frac{t}{|t|}$.
3. $h(x) = \sqrt{1-x^2}$.
4. $s(t) = \sqrt{-t}$.

Solution

1. The given function is

$$f(x) = \frac{x+3}{4-\sqrt{x^2-9}}.$$

The above function is defined everywhere except when

$$\begin{aligned} 4-\sqrt{x^2-9} = 0 &\quad \text{and} \quad x^2-9<0 \\ \Rightarrow x^2-9 = 16 &\quad \text{and} \quad (x-3)(x+3)<0 \\ \Rightarrow x^2-25=0 &\quad \text{and} \quad x \in (-3,3) \\ \Rightarrow (x-5)(x+5)=0 &\quad \text{and} \quad x \in (-3,3) \\ \Rightarrow x = \pm 5 &\quad \text{and} \quad x \in (-3,3). \end{aligned}$$

Thus, the domian of the given function will be

$$D(f) = \mathbb{R} - [(-3,3) \cup \{-5,5\}].$$

2. The given function is

$$g(t) = \frac{t}{|t|}.$$

This function is defined everywhere except when $|t| = 0$, that is, $t = 0$. Thus, the domain will be

$$D(g) = \mathbb{R} - \{0\}.$$

3. The given function is

$$h(x) = \sqrt{1-x^2}.$$

The above function will be defined if

$$1-x^2 \geq 0 \Rightarrow (1-x)(1+x) \geq 0 \Rightarrow x \in [-1,1].$$

Thus the domian will be

$$D(h) = [-1,1].$$

4. The given function is

$$s(t)\sqrt{-t}.$$

Again, this function will be defined if

$$-t \geq 0 \Rightarrow t \leq 0 \Rightarrow t \in (-\infty, 0].$$

Thus, the domian will be

$$D(s) = (-\infty, 0].$$

Problem 6

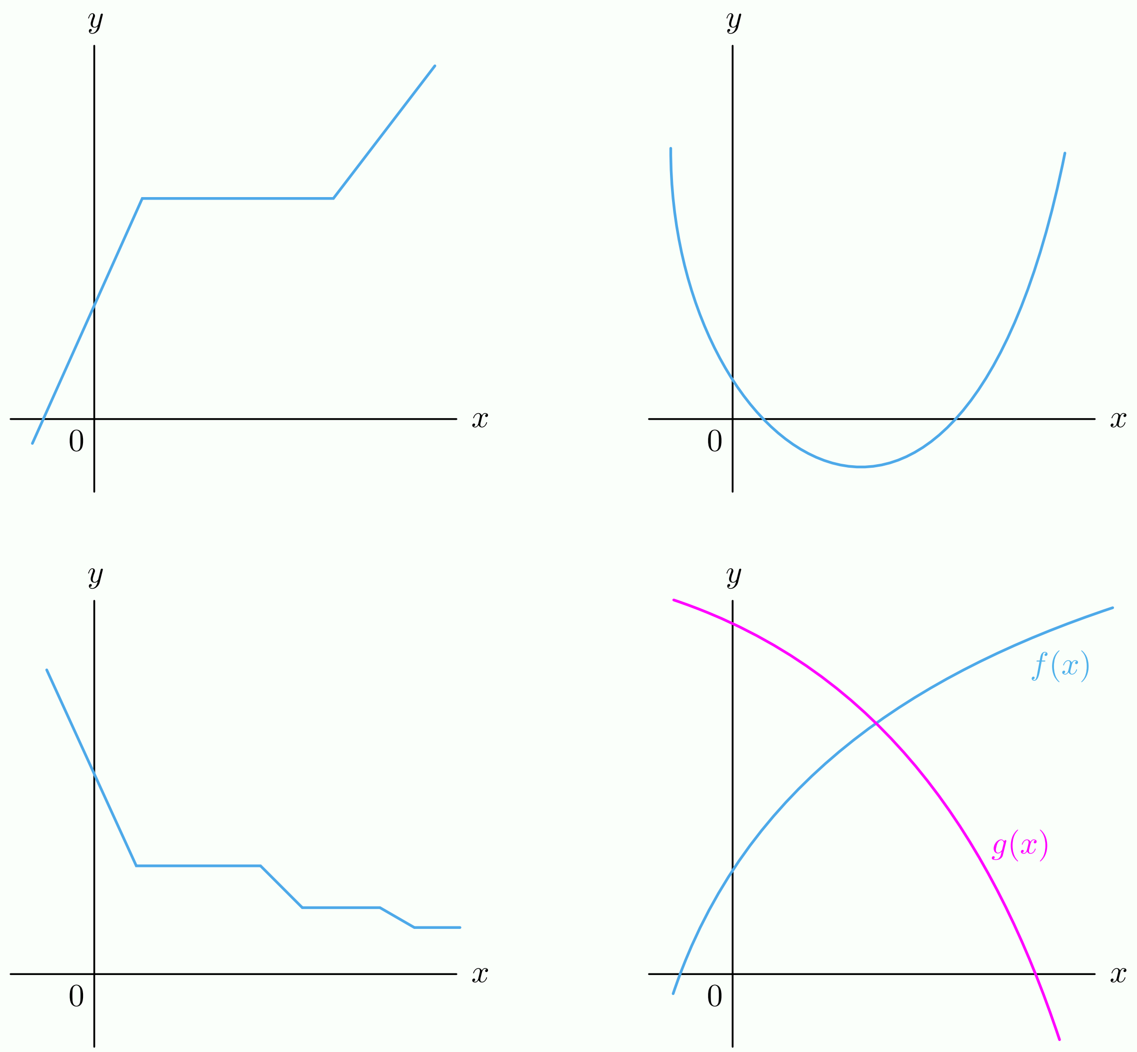
How many points are there in the range of a constant function $f : \mathbb{R} \to \mathbb{R}$?

Solution

Since, a constant function can only take one value, then range contains excatly one point. Thus, there is only one point in the range set.

Problem 7

Write if the functions are **increasing**, **decreasing**, **strictly inccreasing** or **strictly decreasing**.



Solution

- The first function is increasing (**not** strictly increasing).
- The second function is neither increasing nor decreasing.
- The third function is decreasing (**not** strictly decreasing).
- In the last problem, the function f is strictly increasing whereas the function g is strictly decreasing.

Problem 8

Write the function after the given transformations.

1. $f(x) = \sqrt{x}$.
   - Upward 4 units.
   - Right side 10 units.
2. $f(x) = \sin x + \tan x + e^{x^2}$.
   - Towards left 20 units.
   - Downward 5 units.
   - Towards right 20 units.
   - Upward 10 units.

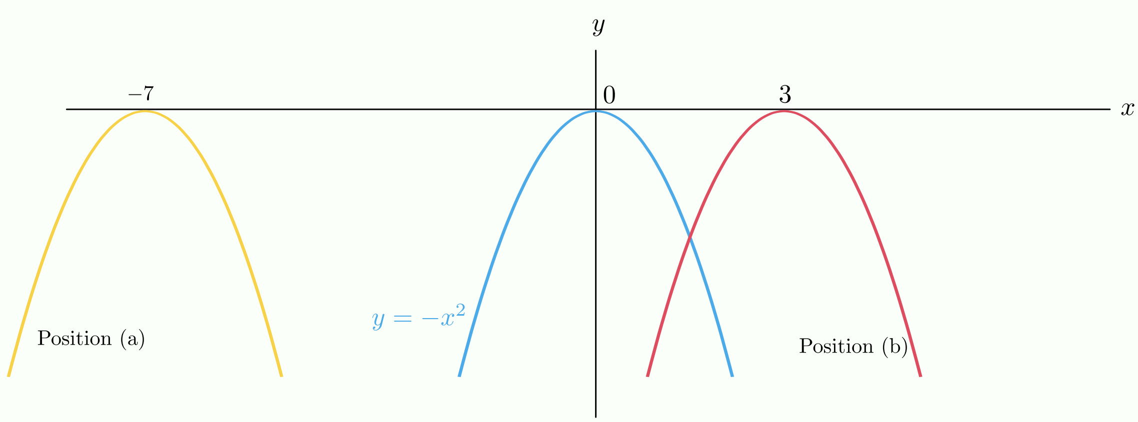
Solution

1. $f(x) = \sqrt{x}$.
   - After first transformation: $F_1(x) = \sqrt{x} + 4$.
   - After second transformation: $F_2(x) = \sqrt{x - 10} + 4$.
2. $f(x) = \sin x + \tan x + e^{x^2}$.
   - After first transformation: $F_1(x) = f(x + 20)$.
   - After second transformation: $F_2(x) = f(x + 20) - 5$.
   - After third transformation: $F_3(x) = f(x + 20 - 20) - 5 = f(x) - 5$.
   - After fourth transformation: $F_4(x) = f(x) - 5 + 20 = f(x) + 15$.

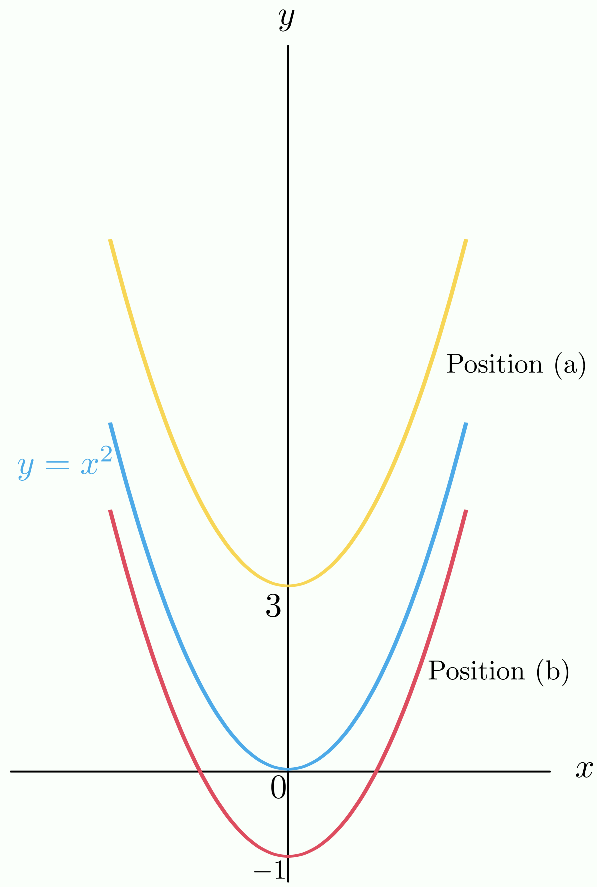
Problem 9

The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.

1.



2.



Solution

1. $f(x) = -x^2$
   - Position (a): $-(x+7)^2$.
   - Position (b): $-(x-3)^2$.
2. $f(x) = x^2$
   - Position (a): $x^2 + 3$.
   - Position (b) = $x^2 - 1$.