Cut locus and Morse-Bott Function

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Top Flavours
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Outline of the talk

1. Morse-Bott Functions
2. Cut locus
   - Cut locus of a point
   - Cut locus of a submanifold
3. An illuminating example
4. Regularity of distance squared function
5. $M - \mathrm{Cu}(N)$ deforms to N

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An Example: Cut locus of south pole in sphere

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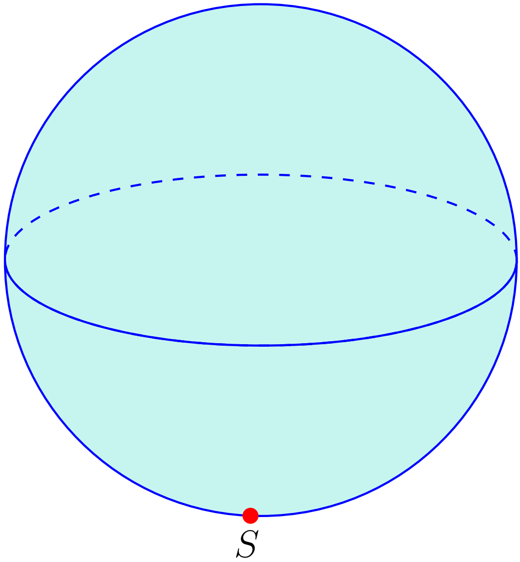

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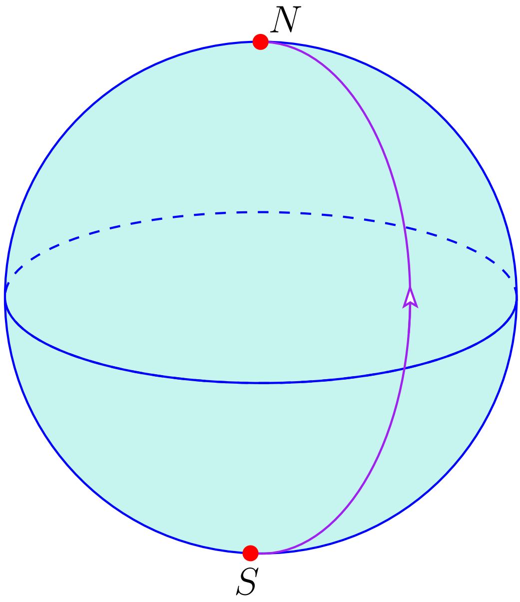

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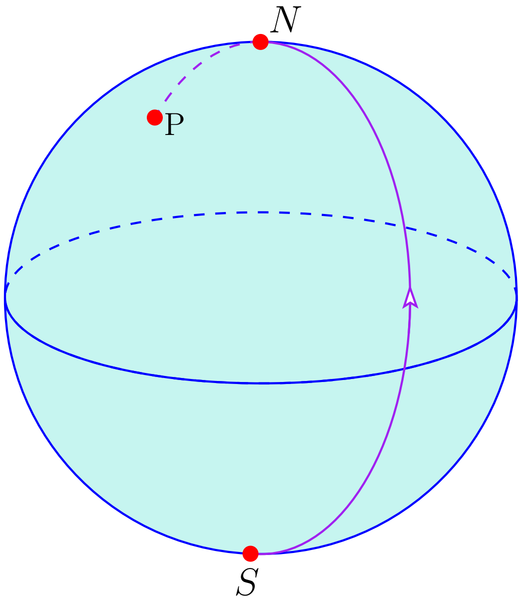

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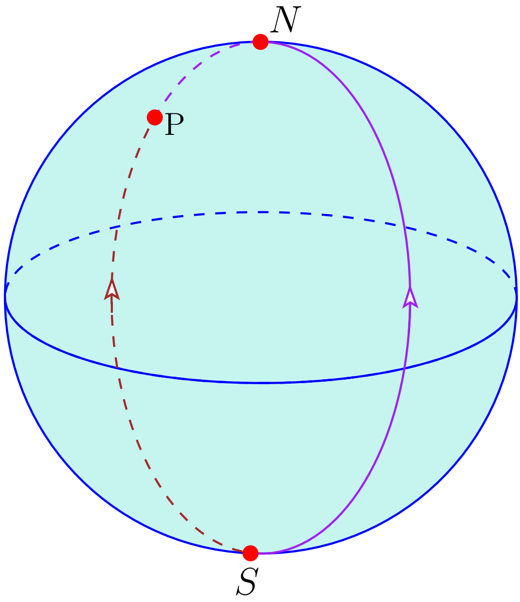

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The cut locus of a sphere in $\mathbb{R}^3$ is its center.

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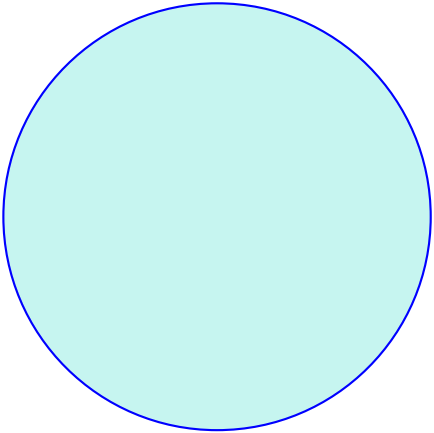

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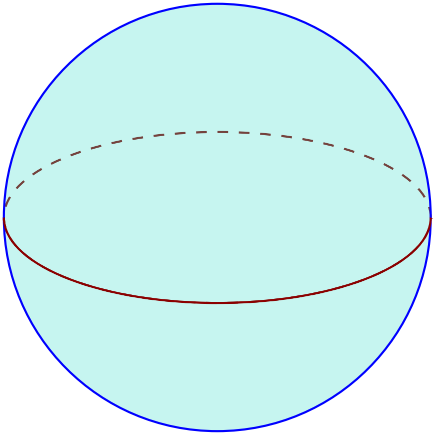

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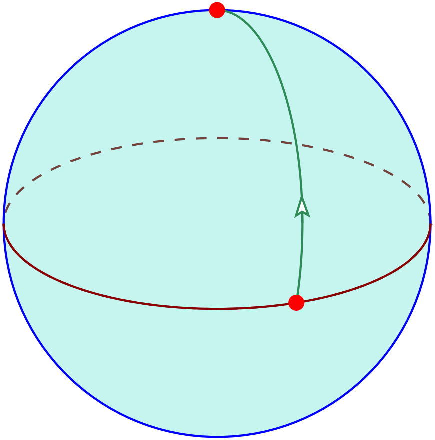

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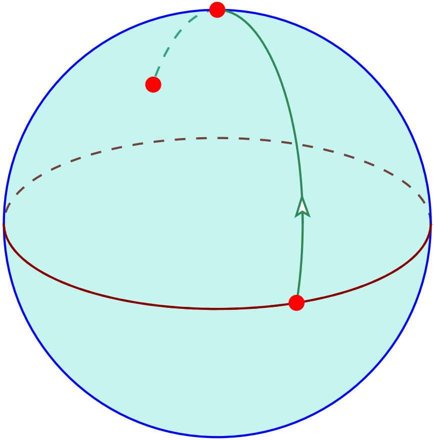

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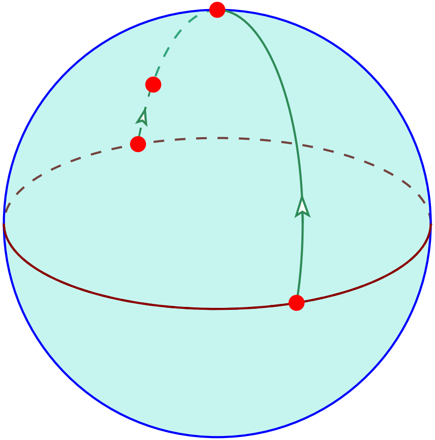

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We will show that f is a Morse-Bott function with critical submanifold as $O(n, \mathbb{R})$.

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Note: The maximizer is unique if and only if A is invertible.

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- The Hessian matrix restricted to $(T_AO(n, \mathbb{R}))^{\perp}$ is $2I_{\frac{n(n+1)}{2}}$.

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$$\gamma(t) = Ae^{-2t} + (1 - e^{-2t})A\left(\sqrt{A^TA}\right)^{-1}. \tag{2}$$

Integral curve of $-\nabla f$

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$$\frac{d\gamma}{dt} = -2\gamma(t) + 2\left(\gamma(t)^T\right)^{-1}\sqrt{\gamma(t)^T\gamma(t)}. \tag{1}$$

- The solution of (1) given by

$$\gamma(t) = Ae^{-2t} + (1 - e^{-2t})A\left(\sqrt{A^TA}\right)^{-1}. \tag{2}$$

- Note that $\gamma(t)$ is a flow line which deforms $GL(n, \mathbb{R})$ to $O(n, \mathbb{R})$.

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Regularity of the distance squared function

Theorem

Let M be a connected, complete Riemannian manifold and N be an embedded submanifold of M. Suppose two N-geodesics exist joining N to $q \in M$. Then $d^2(N, \cdot) : M \to \mathbb{R}$ has no directional direvative ar q for vectors in direction of those two N-geodesics.

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where $S(\nu)$ is the unit normal bundle of N and $[0, \infty]$ is the one-point compactification of $[0, \infty)$.

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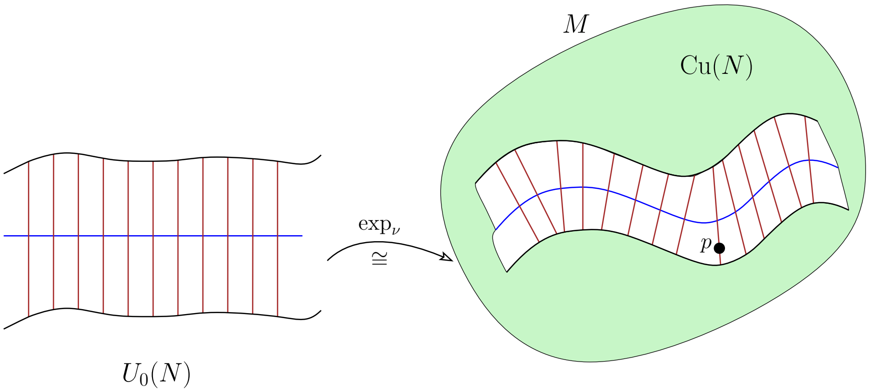
Note that $\exp_\nu$ is a diffeomorphism on $U_0(N)$ and set $U(N) = \exp_\nu(U_0(N)) = M - \mathrm{Cu}(N)$.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$\exp_\nu$
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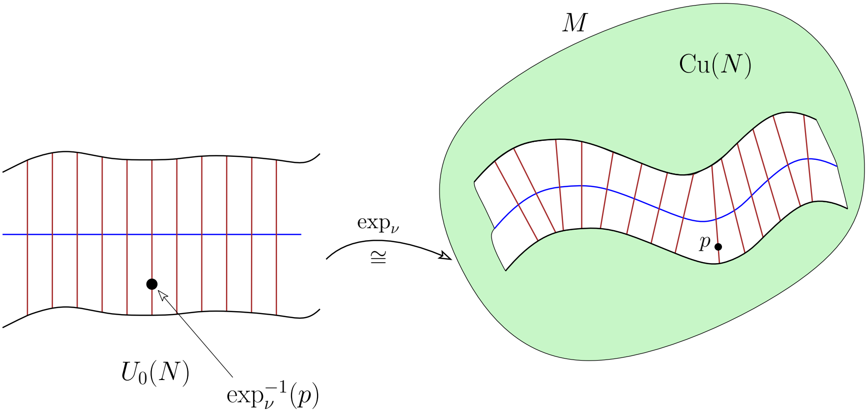

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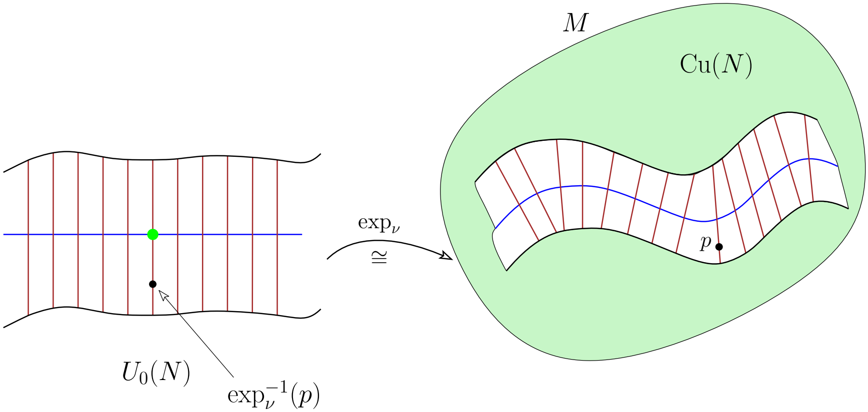
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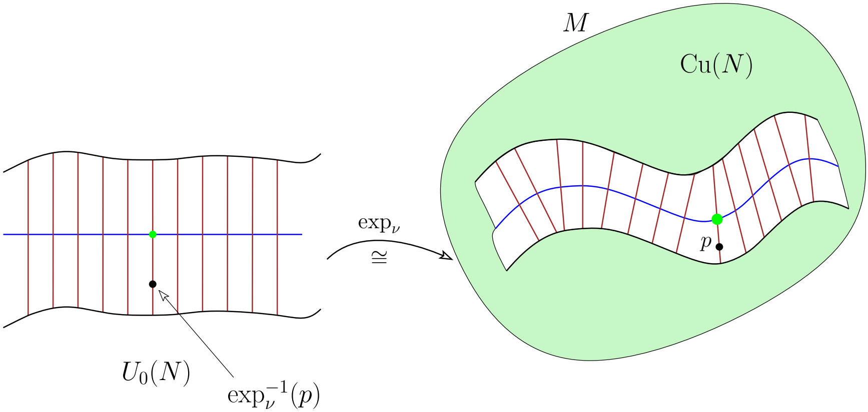
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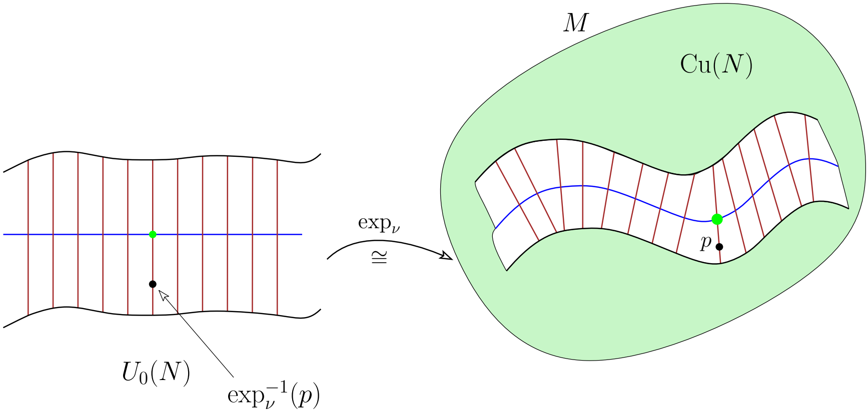
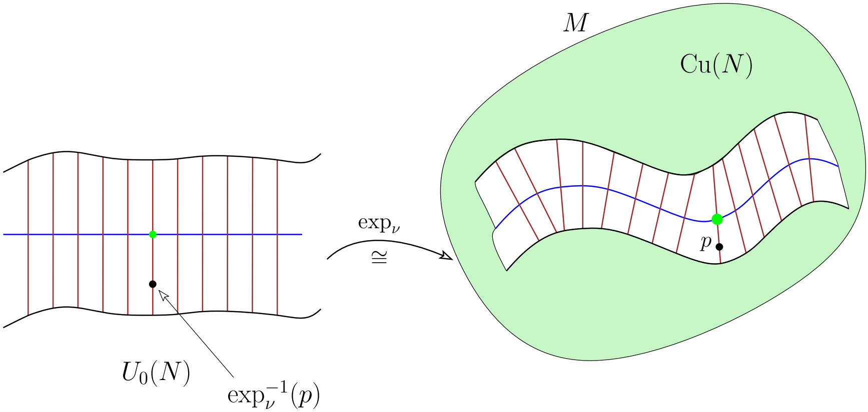


The space $U_0(N)$ deforms to the zero section on the normal bundle.



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$$H : U_0(N) \times [0,1] \to U_0(N), ((p, av), t) \mapsto (p, tav).$$

Now consider the following diagram:

$$\begin{array}{ccc} U_0(N) \times [0,1] & \xrightarrow{H} & U_0(N) \\ \big\uparrow{\scriptstyle \exp_\nu^{-1}} & & \big\downarrow{\scriptstyle exp_\nu} \\ U \times [0,1] & \xrightarrow{F} & U \cong M - \mathrm{Cu}(N) \end{array}$$

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The map F can be defined by taking the compositions

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□

We saw that for $M = M(n, \mathbb{R})$ and $N = O(n, \mathbb{R})$, the cut locus $\mathrm{Cu}(O(n, \mathbb{R}))$ is the set of all singular matrices and $M - \mathrm{Cu}(O(n, \mathbb{R}))$, which is the set of invertible matrices, deforms to $O(n, \mathbb{R})$.

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