Cut locus and Morse-Bott Function

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Outline of the talk

Morse Functions

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An Example: Cut locus of south pole in sphere

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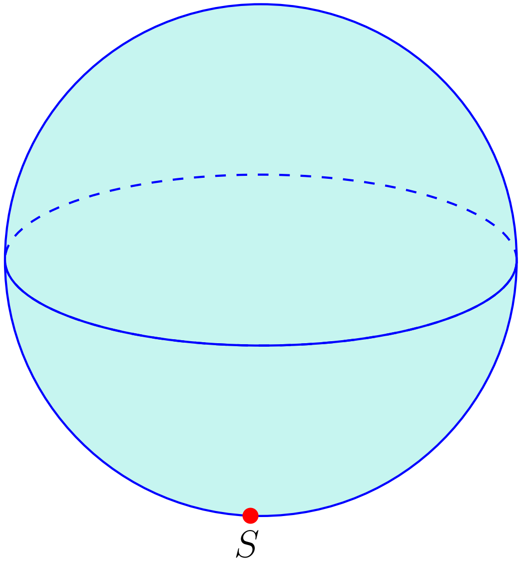

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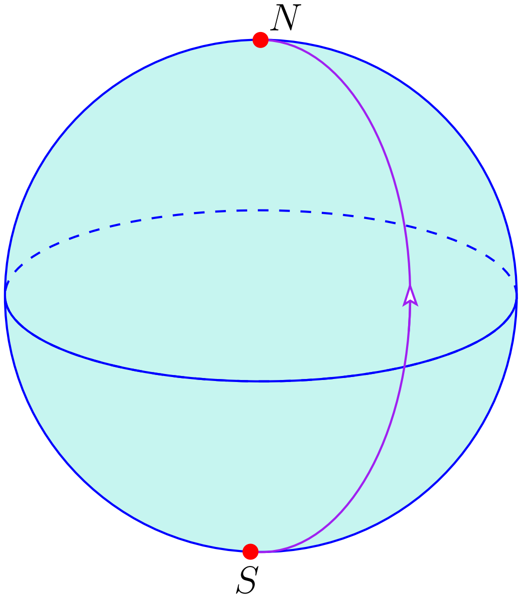

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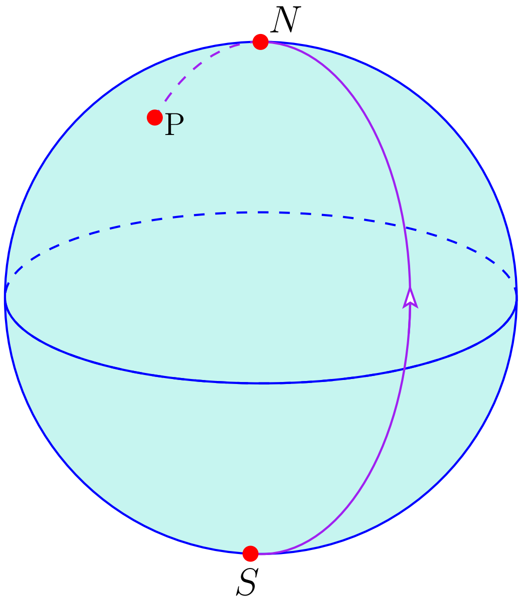

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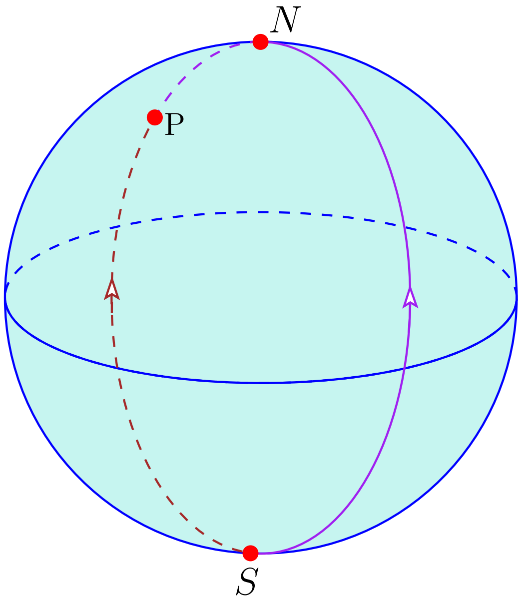

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The cut locus of a sphere in $\mathbb{R}^3$ is its center.

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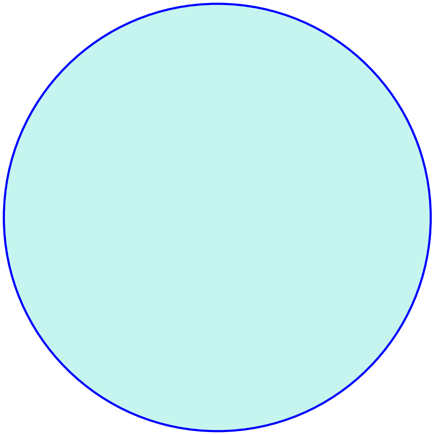

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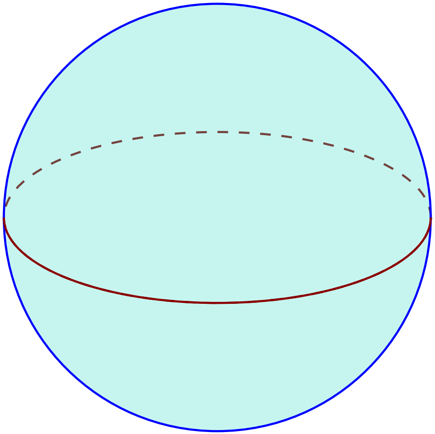

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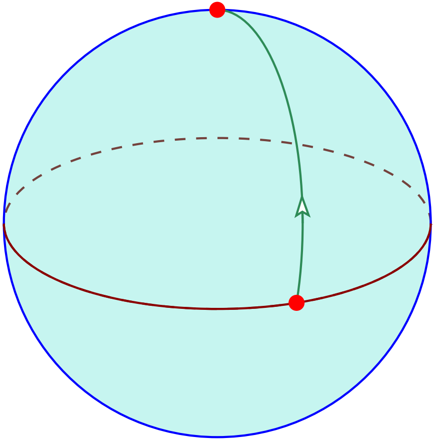

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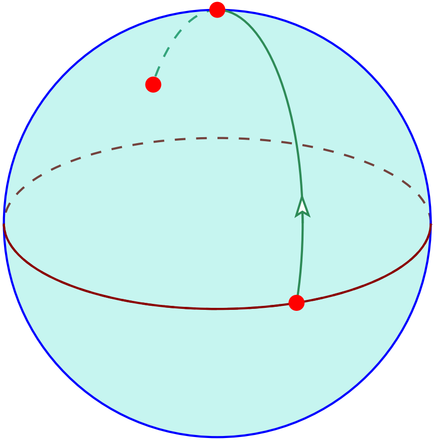

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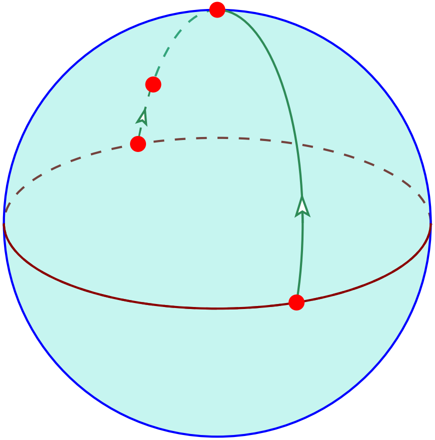

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We will show that f is a Morse-Bott function with critical submanifold as $O(n, \mathbb{R})$.

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Note: The maximizer is unique if and only if A is invertible.

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- $B \in (T_A O(n, \mathbb{R})^{\perp}$ if $B = AW$ for some symmetric matrix W.
- The Hessian matrix restricted to $(T_A O(n, \mathbb{R}))^{\perp}$ is $2I_{\frac{n(n+1)}{2}}$.

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- Note that $\gamma(t)$ is a flow line which deforms $GL(n, \mathbb{R})$ to $O(n, \mathbb{R})$.

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Regularity of the distance squared function

Theorem

Let M be a connected, complete Riemannian manifold and N be an embedded submanifold of M. Suppose two N-geodesics exist joining N to $q \in M$. Then $d^2(N, \cdot) : M \to \mathbb{R}$ has no directional derivative at q for vectors in direction of those two N-geodesics.

Outline of the proof

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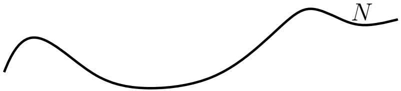

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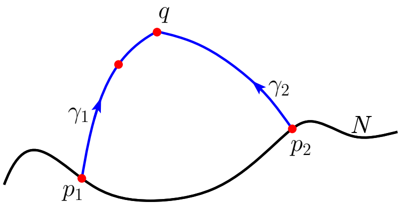

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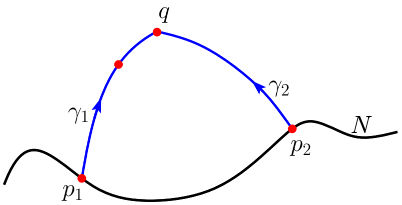

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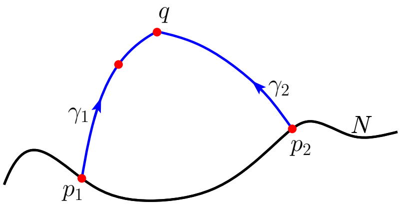

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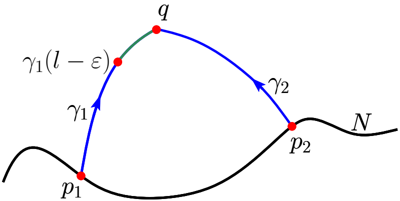

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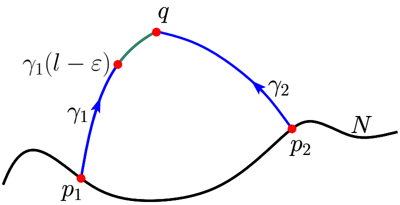

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$$(d^2)'_-(q) := \lim_{\varepsilon \to 0^+} \frac{(d(N, \gamma_i(l)))^2 - (d(N, \gamma_i(l-\varepsilon)))^2}{\varepsilon}$$

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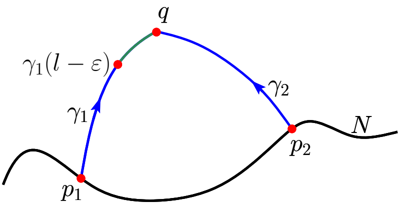

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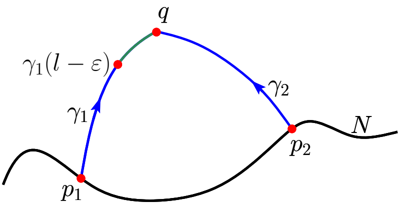

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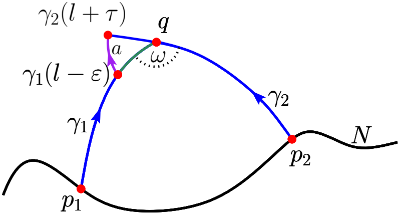

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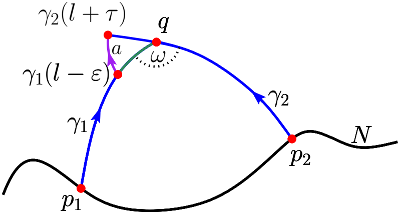

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$$a^2 = \varepsilon^2 + \tau^2 + 2\varepsilon\tau\cos\omega + K(\tau)\varepsilon^2\tau^2$$

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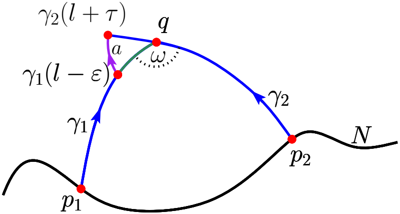

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- Finally, we can show that the derivative from the right is strictly bounded above by $2l$.

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Proposition

Consider the distance squared function with respect to a submanifold N in M. Then this is a Morse-Bott function with N as the critical submanifold.

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where $S(\nu)$ is the unit normal bundle of N and $[0, \infty]$ is the one-point compactification of $[0, \infty)$.

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$$\mathrm{Cu}(N) = \exp_\nu \{\mathbf{s}(v)v : v \in S(\nu)\},$$

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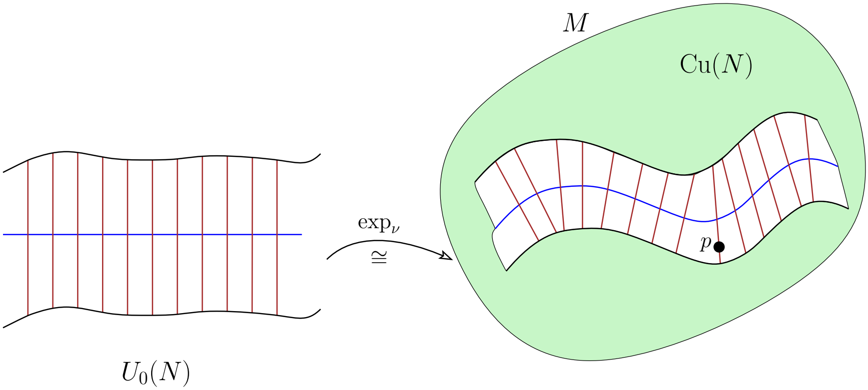
Note that $\exp_\nu$ is a diffeomorphism on $U_0(N)$ and set $U(N) = \exp_\nu(U_0(N)) = M - \mathrm{Cu}(N)$.



$U_0(N)$
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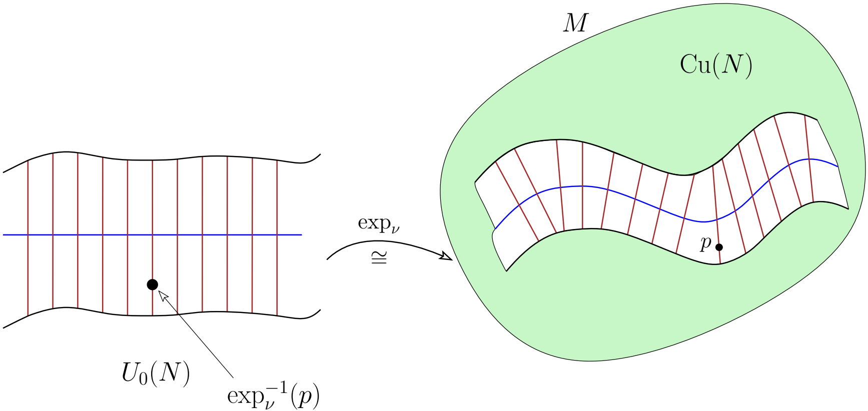
$\cong$
M
$\mathrm{Cu}(N)$
p



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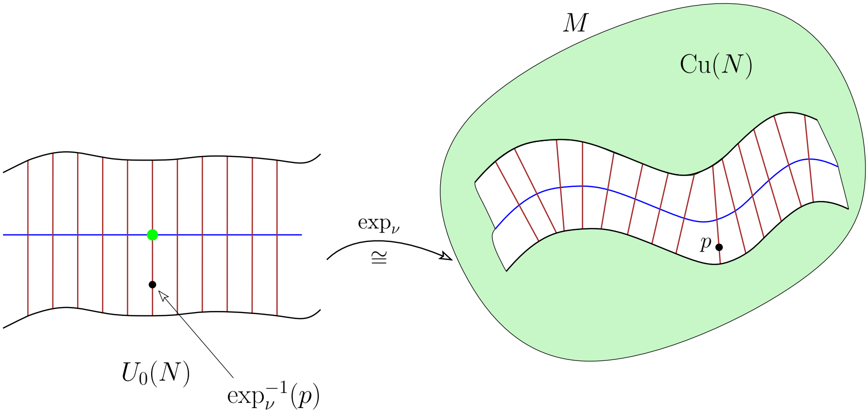

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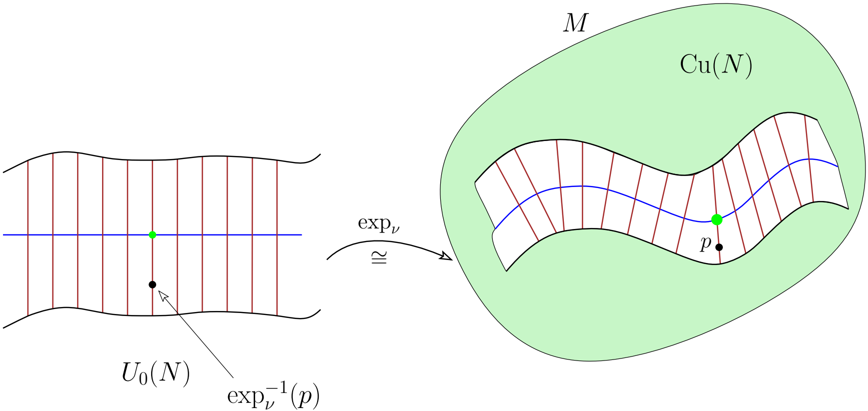

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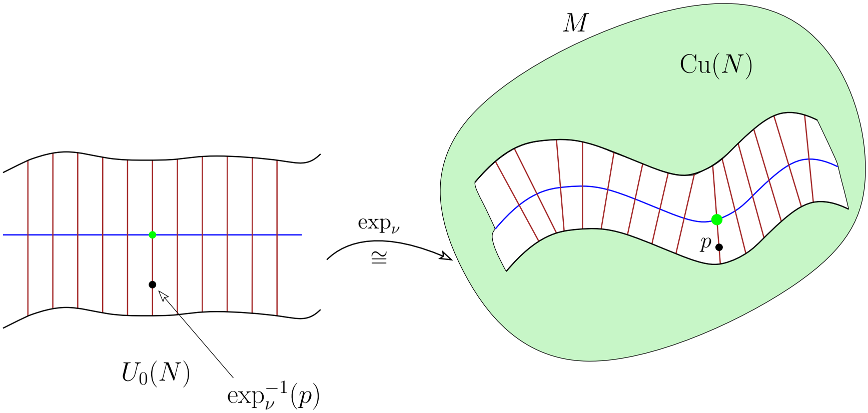
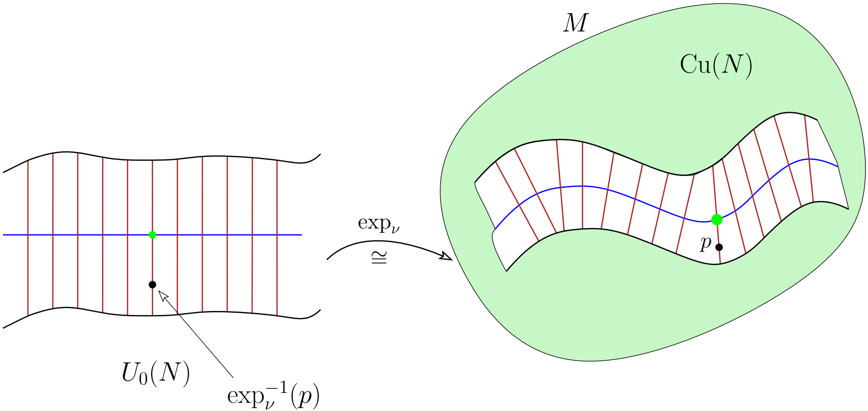


The space $U_0(N)$ deforms to the zero section on the normal bundle.



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$$H : U_0(N) \times [0,1] \to U_0(N), ((p, av), t) \mapsto (p, tav).$$

Now consider the following diagram:

$$\begin{tikzcd}
U_0(N) \times [0,1] \arrow[r, "H"] & U_0(N) \arrow[d, "exp_\nu"] \\
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The map F can be defined by taking the compositions

$$F = \mathrm{exp}_\nu \circ H \circ \mathrm{exp}_\nu^{-1}.$$

$\square$

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$$\begin{array}{ccc} U_0(N) \times [0,1] & \xrightarrow{\quad H \quad} & U_0(N) \\ \uparrow {\scriptstyle \exp_\nu^{-1}} & & \downarrow {\scriptstyle exp_\nu} \\ U \times [0,1] & \xrightarrow{\quad F \quad} & U \cong M - \mathrm{Cu}(N) \end{array}$$

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□

We saw that for $M = M(n, \mathbb{R})$ and $N = O(n, \mathbb{R})$, the cut locus $\mathrm{Cu}(O(n, \mathbb{R}))$ is the set of all singular matrices and $M - \mathrm{Cu}(O(n, \mathbb{R}))$, which is the set of invertible matrices, deforms to $O(n, \mathbb{R})$.

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