Equivariant Cut Locus Theorem

Inter IISER-NISER Mathematics Meet, 2022

IISER Kolkata

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Indian Institute of Science Education and Research Kolkata

1st June 2022

IISER KOLKATA

Outline of the talk

1. Statement of the main result
2. Idea of the proof
3. Geodesics on M and M/G
4. Proof of the main theorem

Statement of the main result

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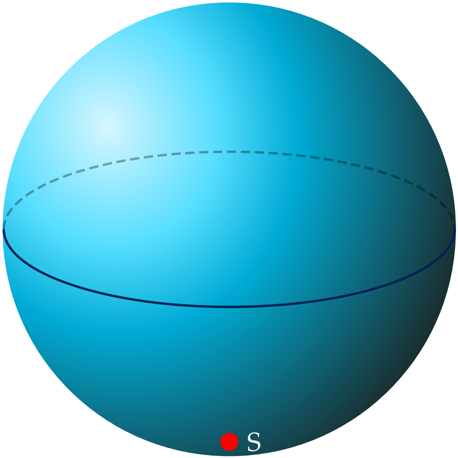

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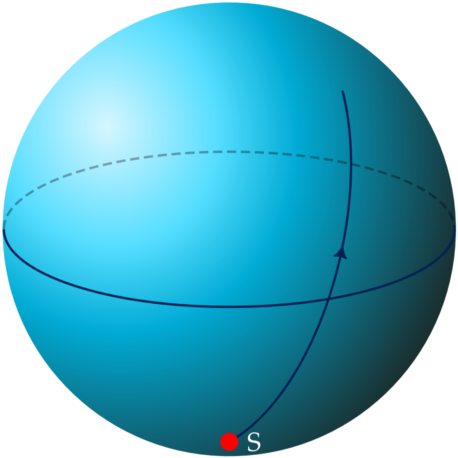

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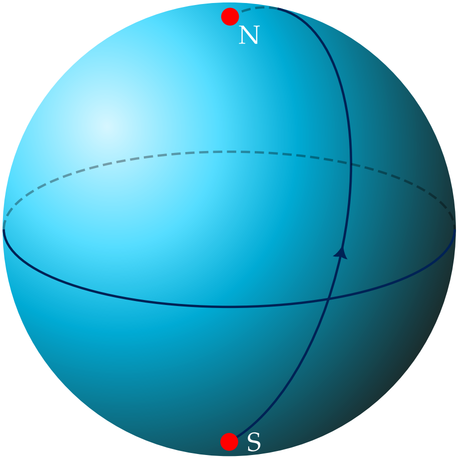

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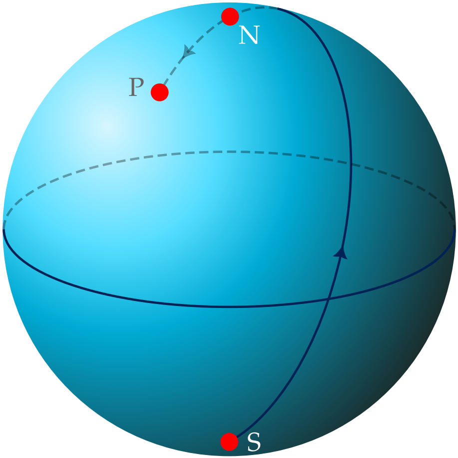

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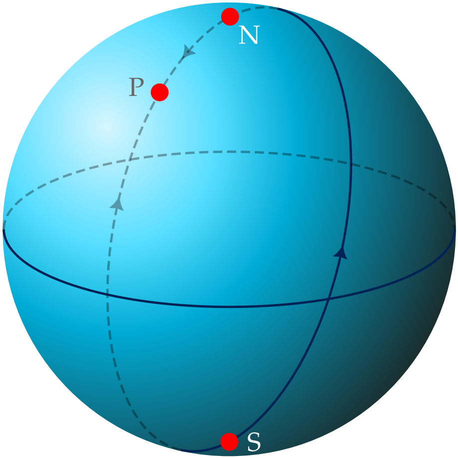

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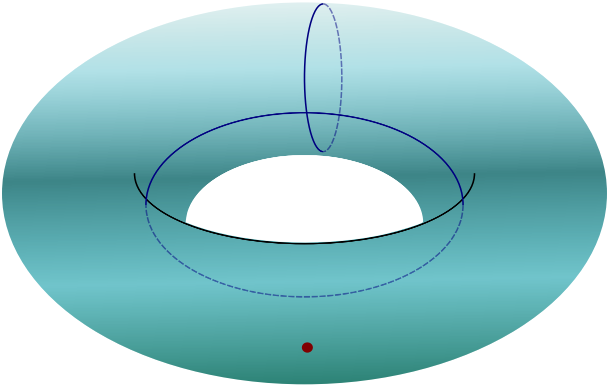

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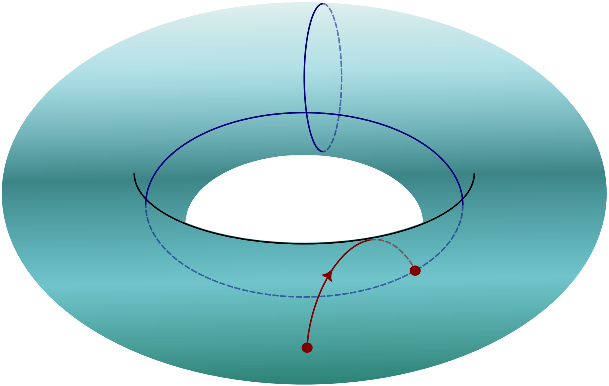

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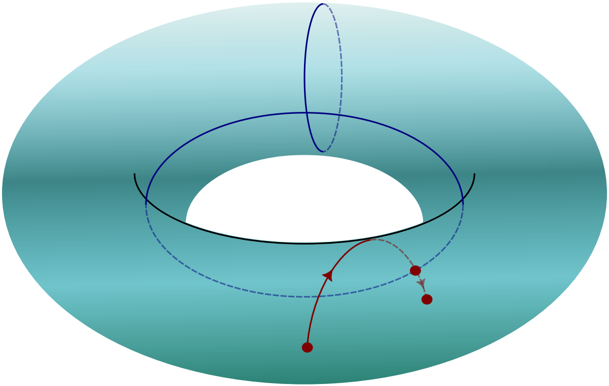

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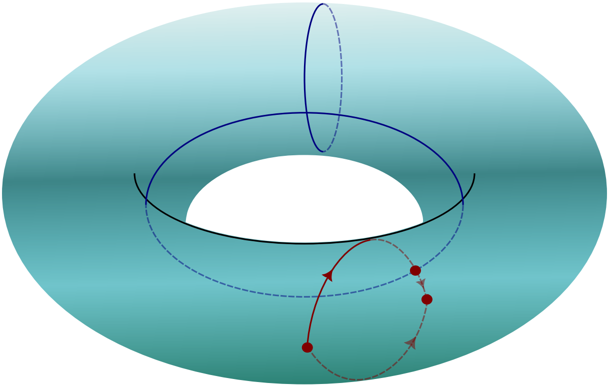

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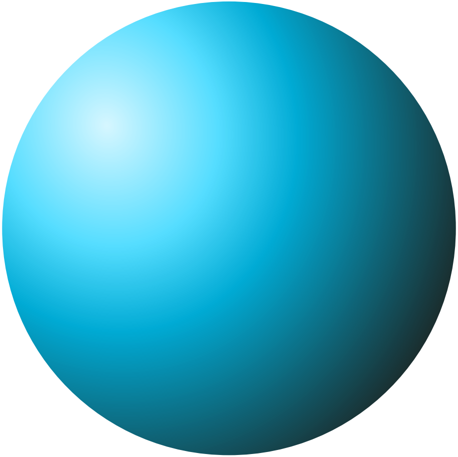

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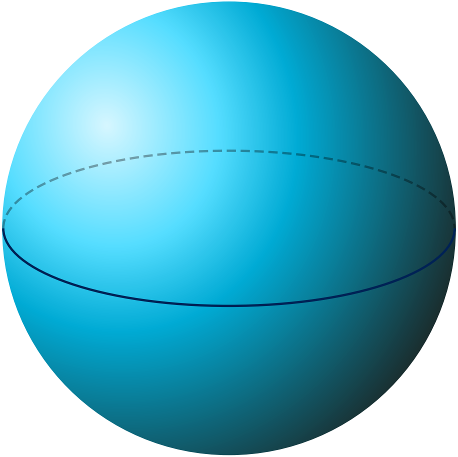

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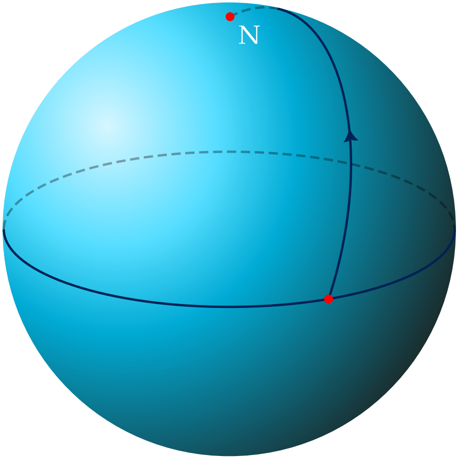

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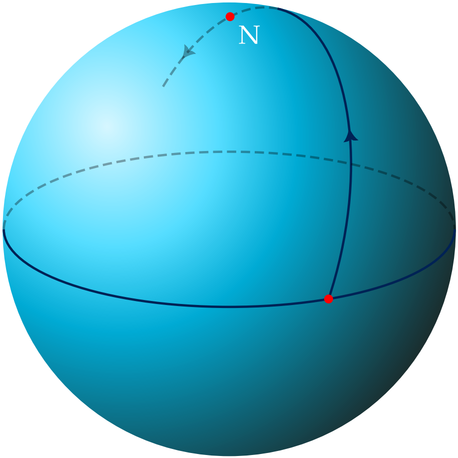

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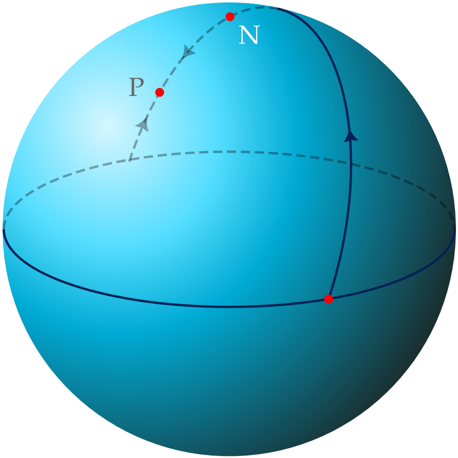

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Theorem

For a complete Riemannian manifold M and a compact submanifold N of M,

$$\overline{\mathrm{Se}(N)} = \mathrm{Cu}(N).$$

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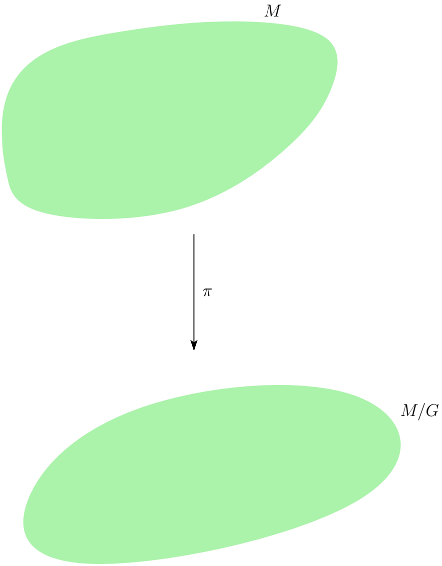

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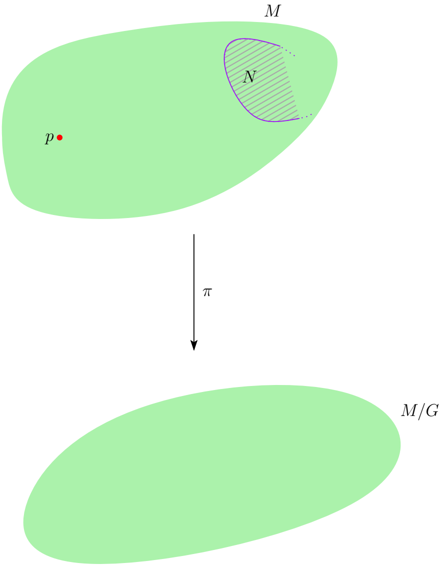

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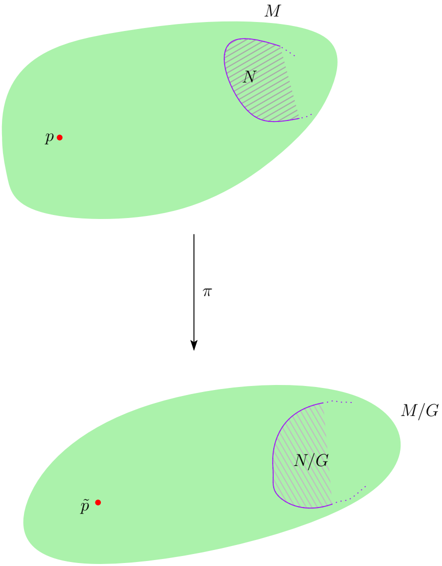

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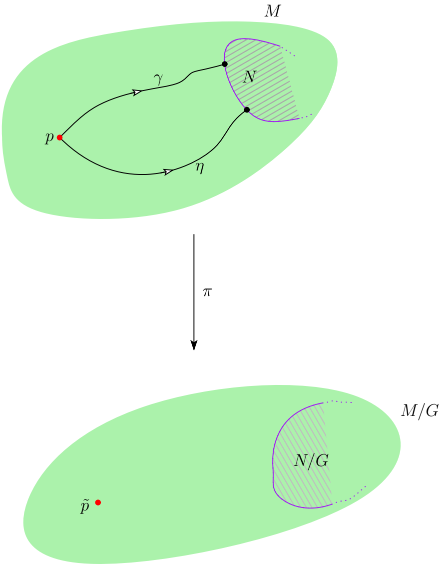

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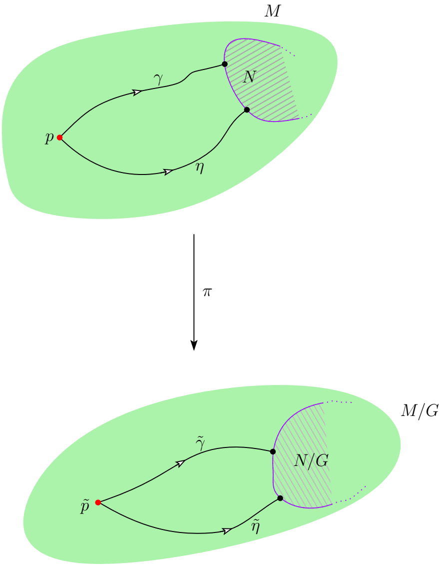

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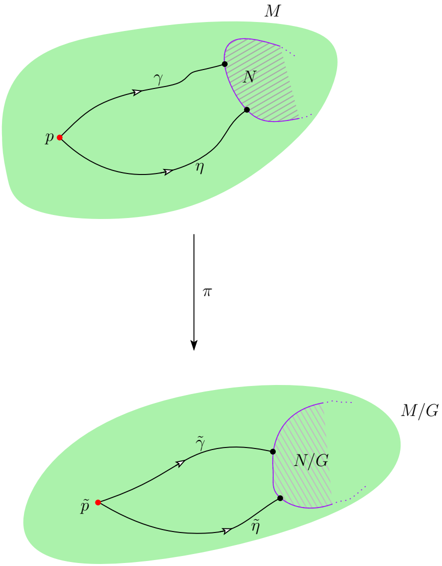

Problems in the approach

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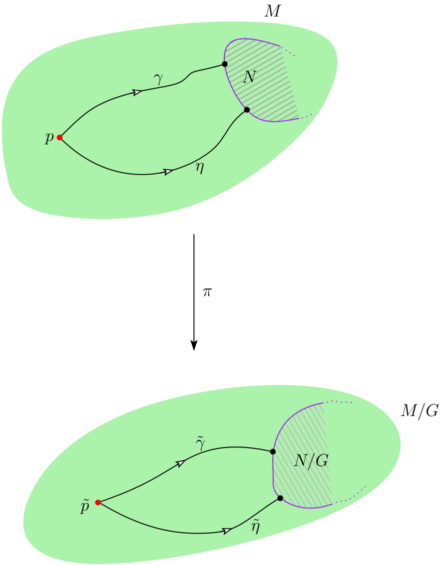

Problems in the approach

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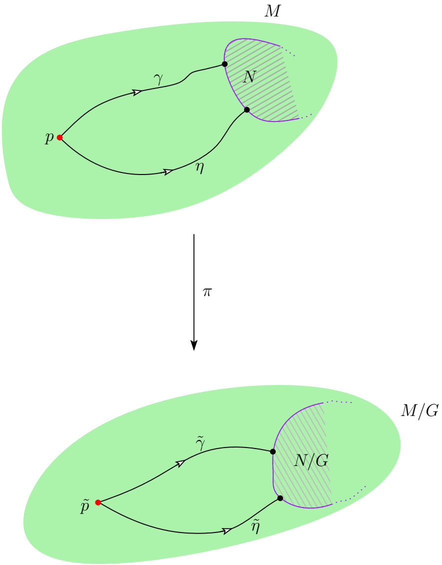

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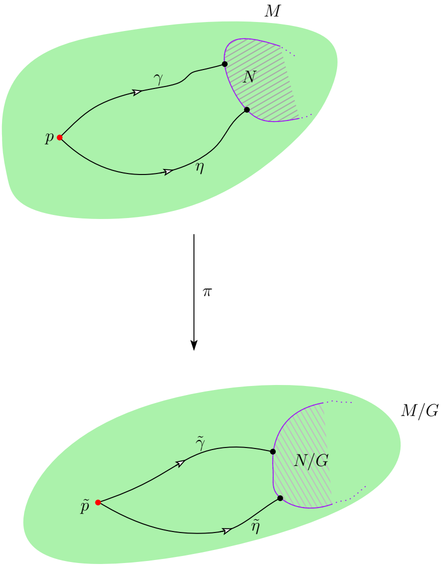

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3. The same for the lifts.

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- $d\pi$ restricts to $\mathcal{H}_p$ is an isomorphism on $T_{\pi(p)}B$.
- dg maps $\mathcal{H}_p$ to $\mathcal{H}_{g \cdot p}$.

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Proposition (Uniqueness of horizontal lift)

If $\gamma : [-1, 1] \to B$ is a smooth curve such that $\gamma(0) = b$ and $e_0 \in \pi^{-1}(b)$, then there is a unique horizontal lift $\tilde{\gamma}$ through $e_0 \in E$.

Geodesics on M and M/G

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Theorem

There is a one-to-one correspondence between the geodesics on M/G and geodesics on M which are horizontal.

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Lemma

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If $\tilde{\gamma}$ is a geodesic on E and $\tilde{\gamma}'(0) \in \mathcal{H}_{\gamma(0)}$, then for all t, $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ and $\pi \circ \gamma$ is a geodesic on B. Moreover, the length is preserved.

Does the horizontal lift $\tilde{\gamma}$ of a geodesic γ become a geodesic?

Lemma

Let γ be a geodesic in B. If $\tilde{\gamma}$ is the horizontal lift of γ, then

- $l(\gamma) = l(\tilde{\gamma})$.

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Since $\pi : M \to M/G$ is Riemannian submersion, we will get a one-to-one correspondence between the geodesics on M/G and geodesics on M which are horizontal.

Proof of the main theorem

$\mathrm{Se}(N)/G \subseteq \mathrm{Se}(N/G)$

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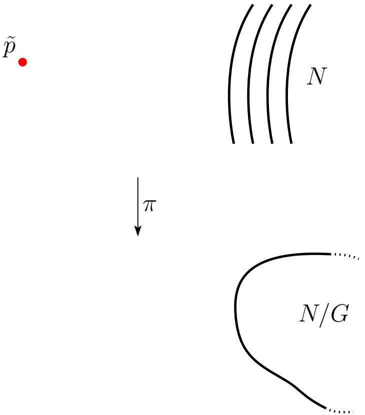

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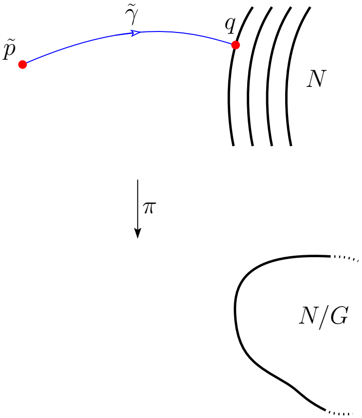

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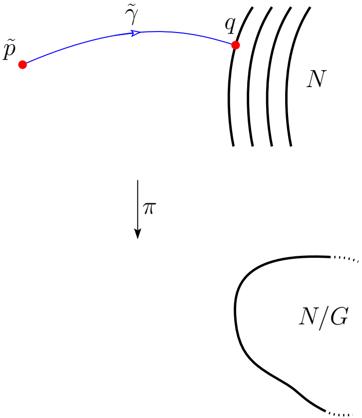

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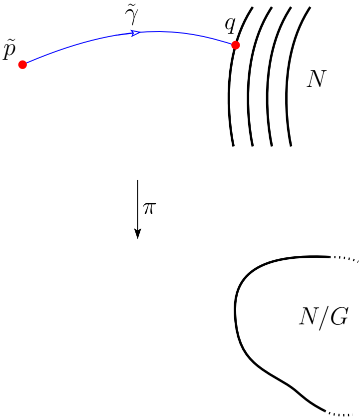

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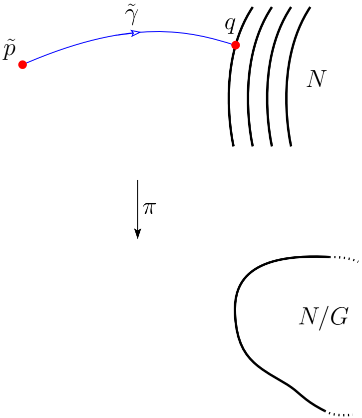

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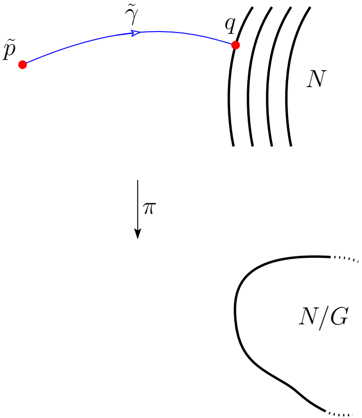

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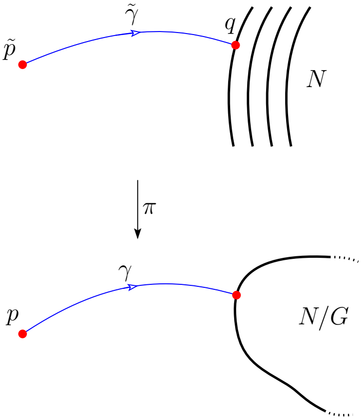

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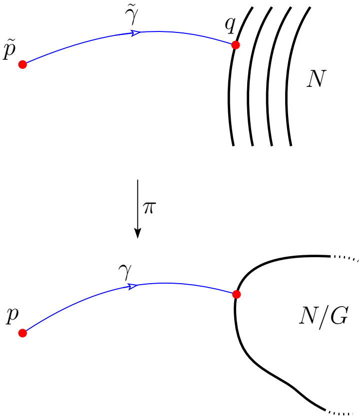

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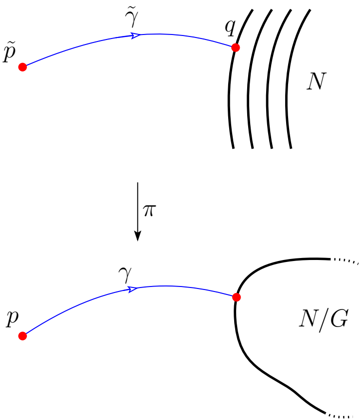

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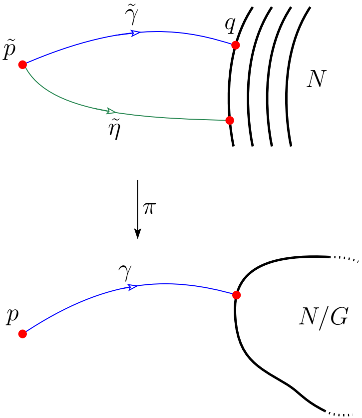

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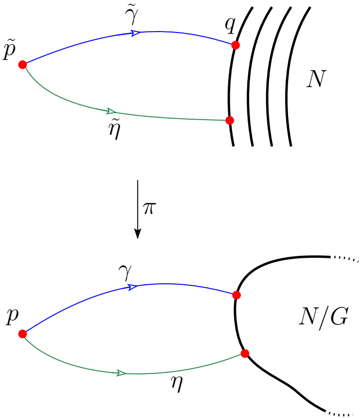

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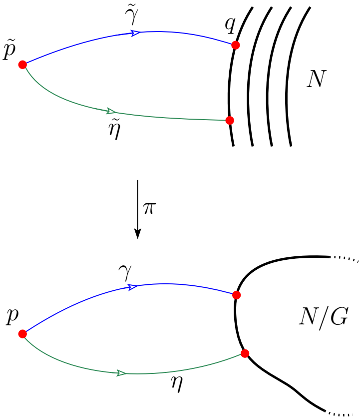

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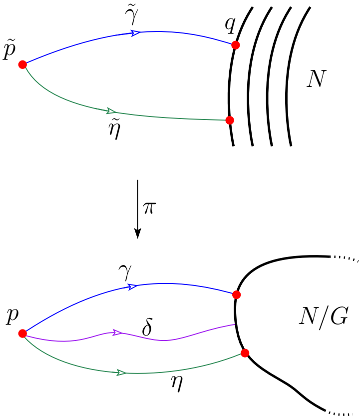

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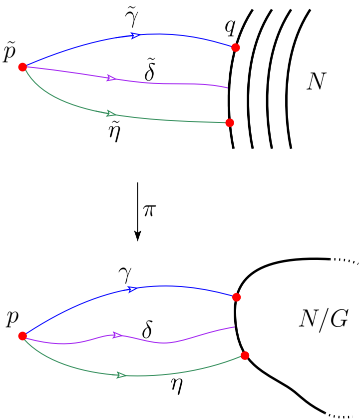

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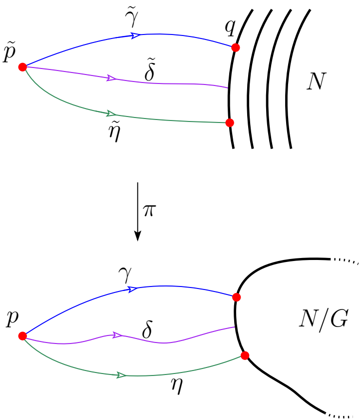

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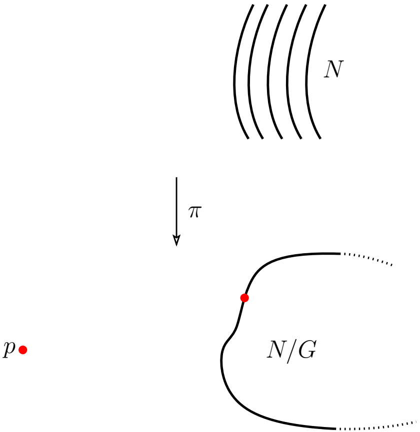

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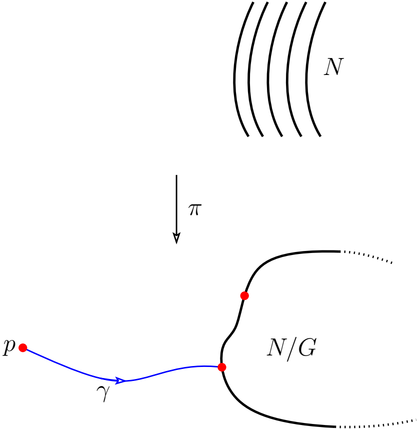

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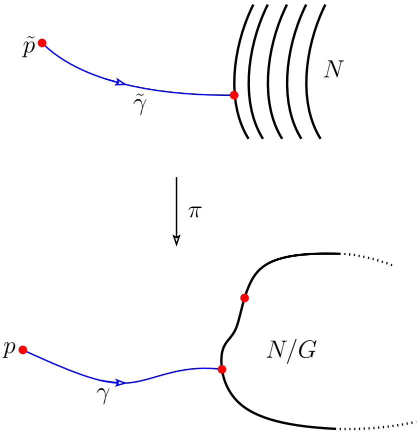

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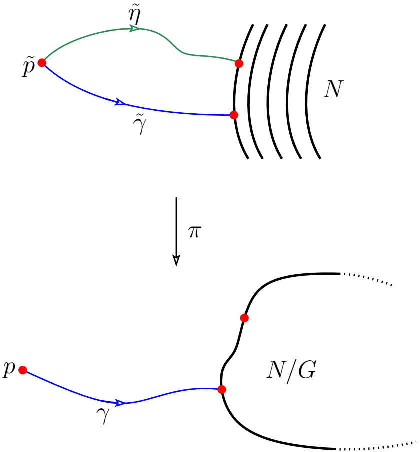

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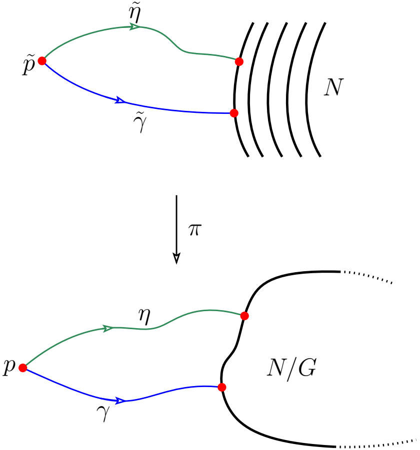

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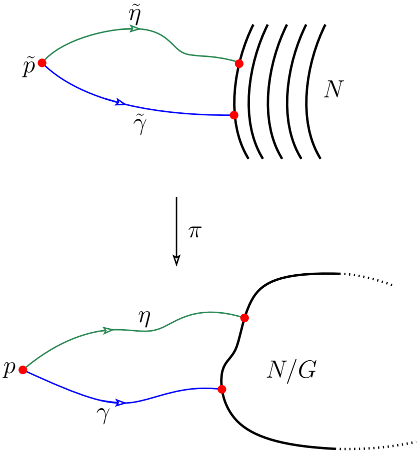

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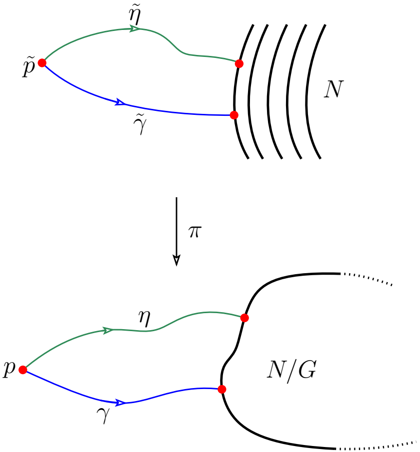
$\mathrm{Se}(N)/G \supseteq \mathrm{Se}(N/G)$



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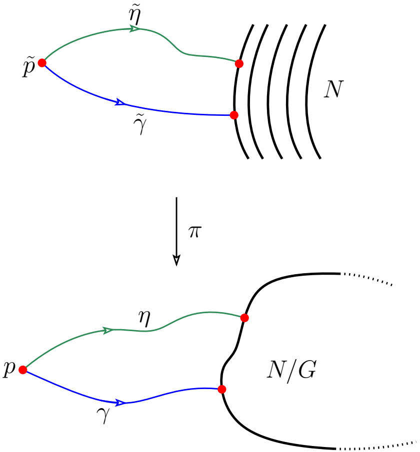
$\mathrm{Se}(N)/G \supseteq \mathrm{Se}(N/G)$



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l(\gamma) &= d(p, N/G) = l(\eta) \\
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\end{aligned}
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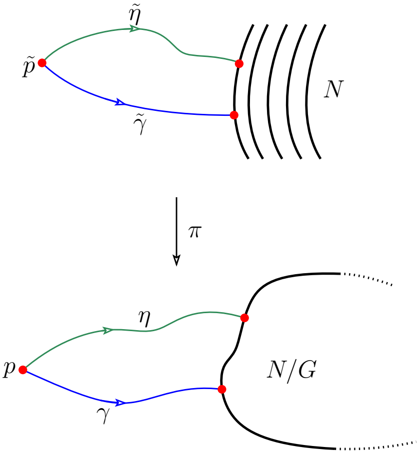

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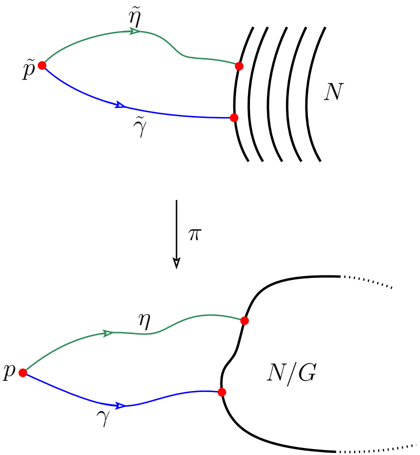

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a contradiction.

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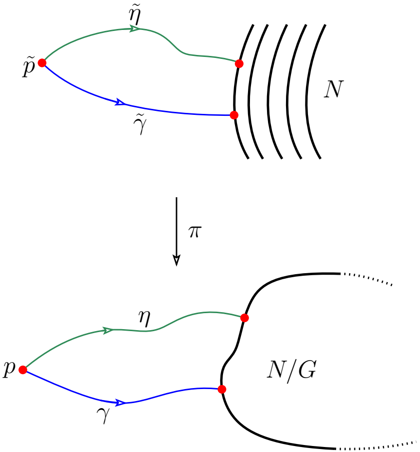

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- Thus, $\tilde{p} \in \mathrm{Se}(N)$.

Thank you for your attention!