# Independence Complex of Wedge of Graphs

ICTS In-House 2023

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International Centre for Theoretical Sciences Bangalore

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ICTS INTERNATIONAL CENTRE *for* THEORETICAL SCIENCES

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

# Background

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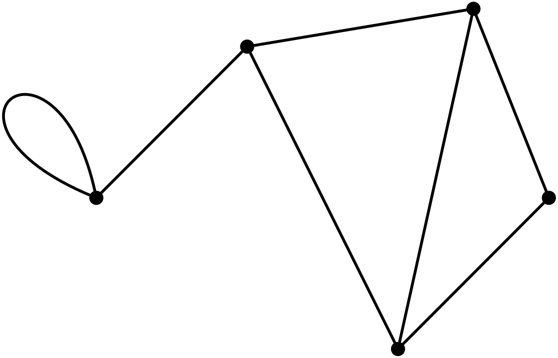

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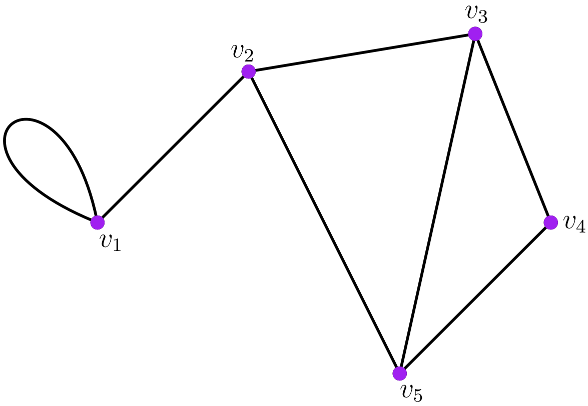

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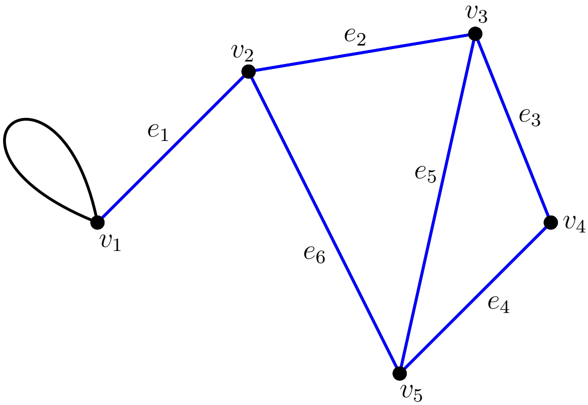

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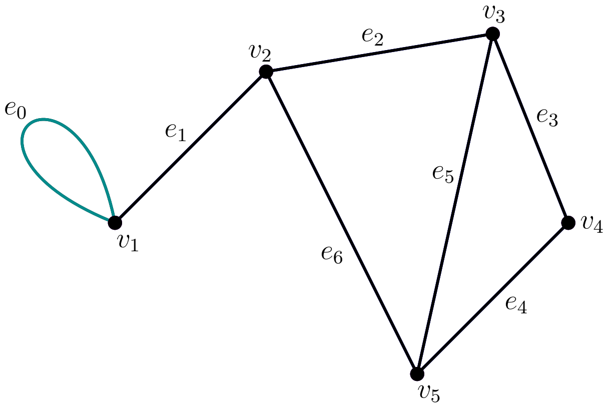

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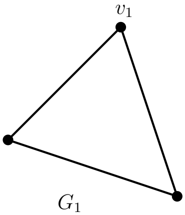

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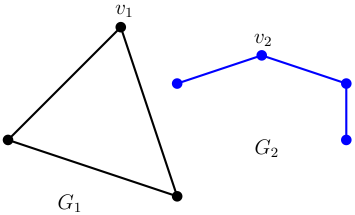

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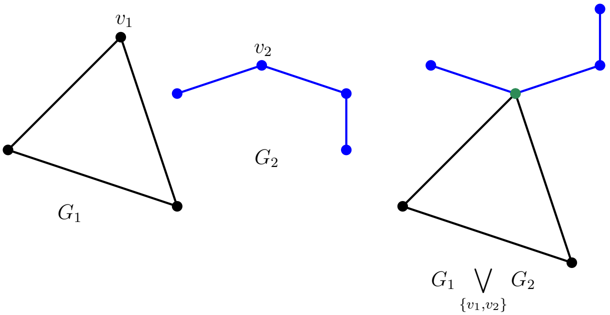

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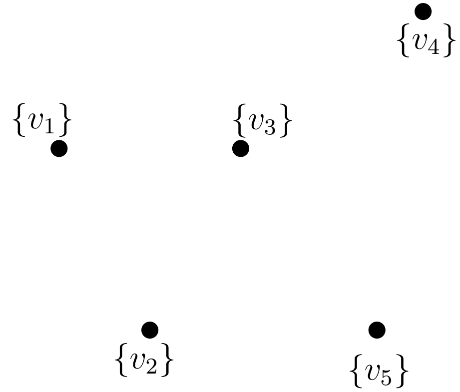

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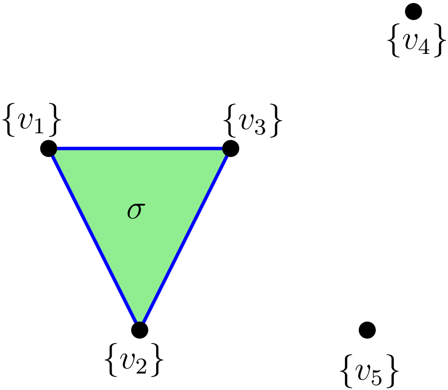

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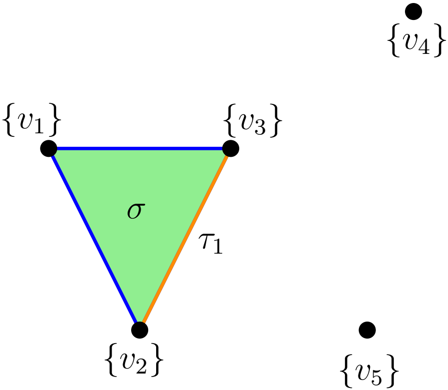

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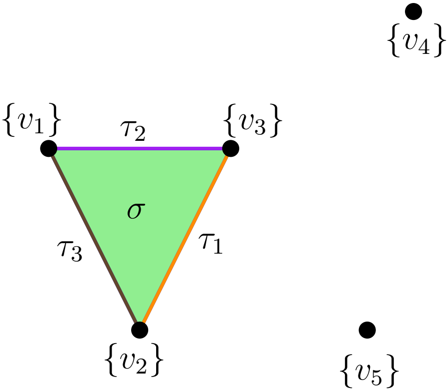

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For a finite simple graph $G$, with the vertex set $V$, the independence complex $\mathcal{I}(G)$ is the simplicial complex consisting of all independent subsets of $V$ (i.e. no two vertices are adjacent) as its simplices.

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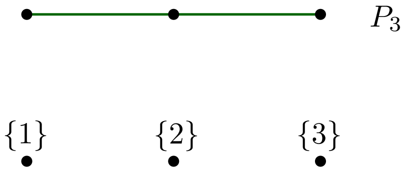

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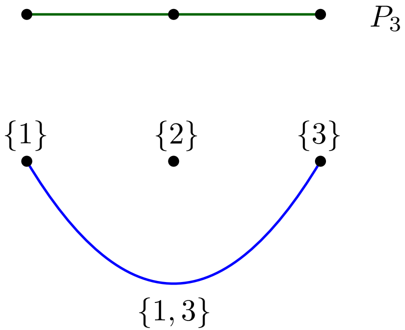

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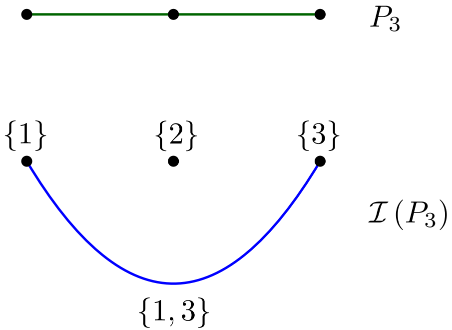

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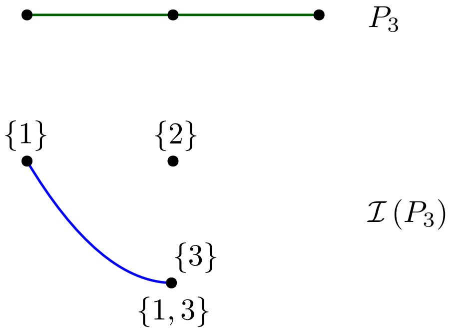

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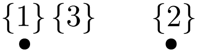


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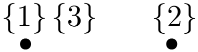


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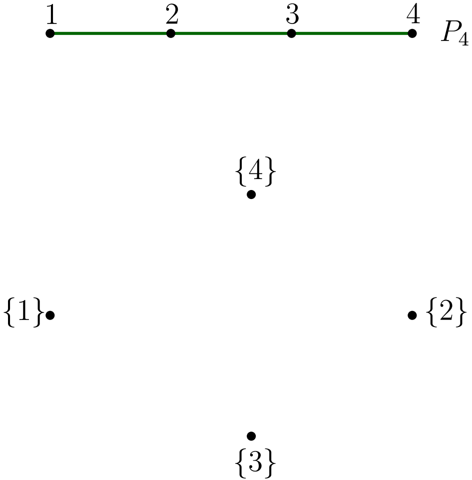

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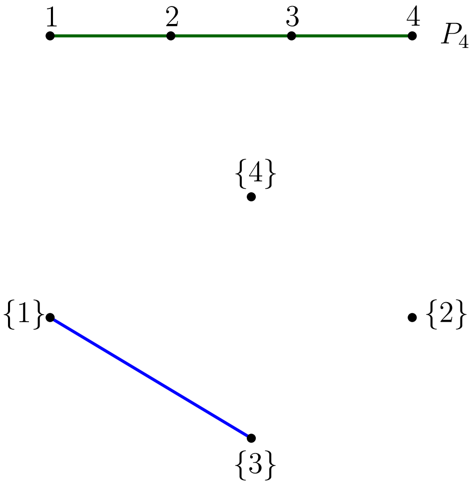

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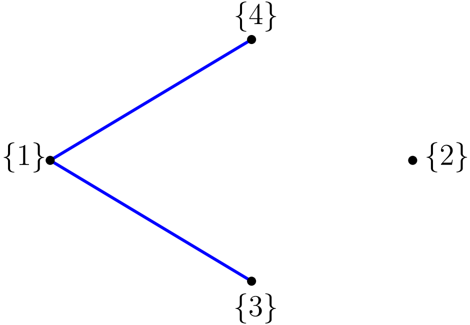

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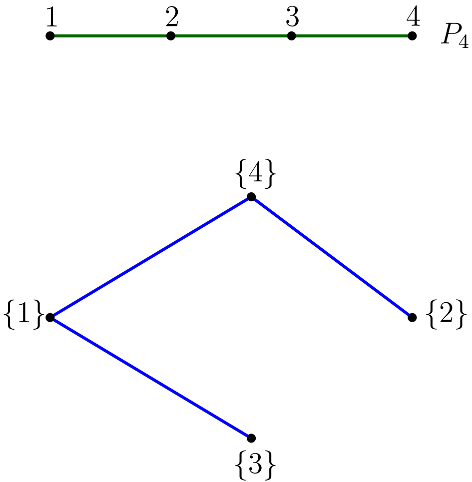

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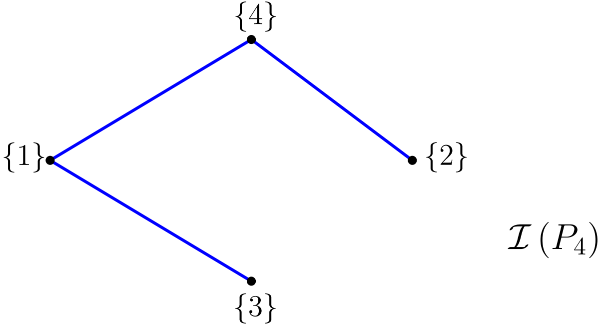

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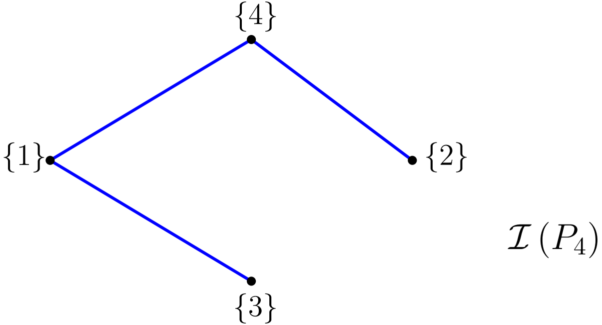


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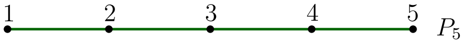

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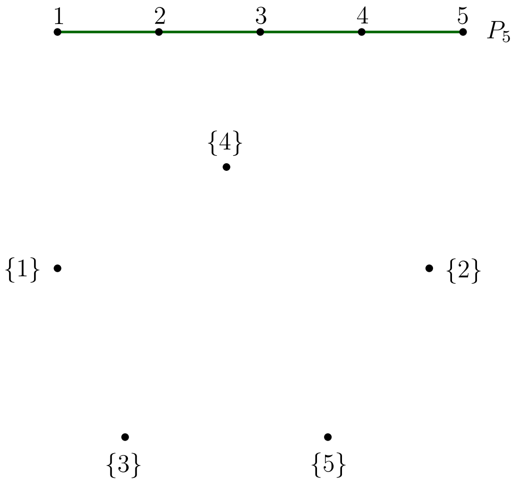

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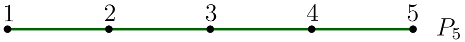


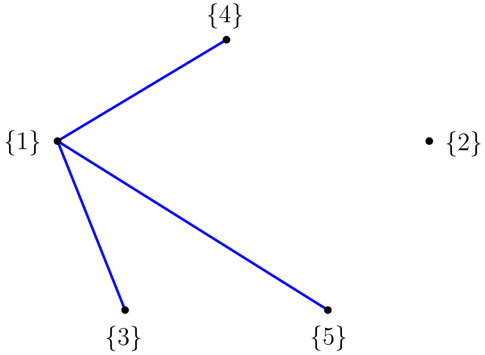

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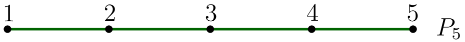


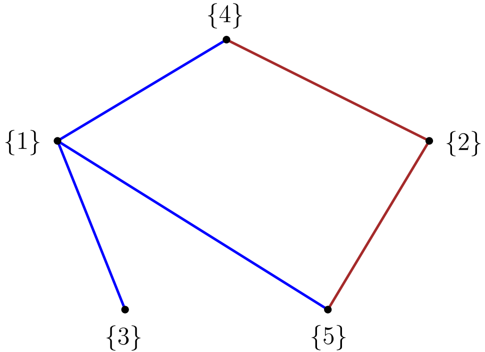

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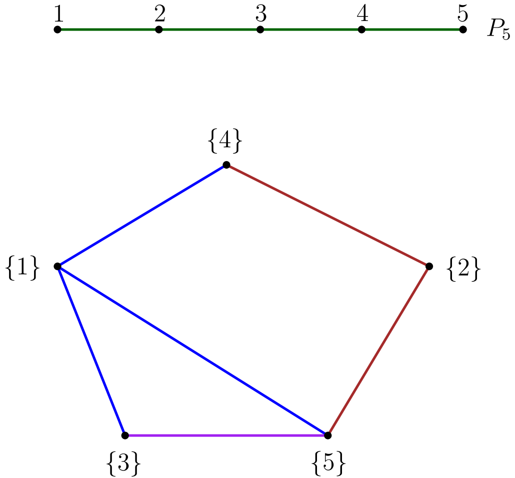

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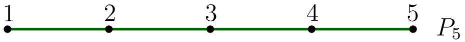


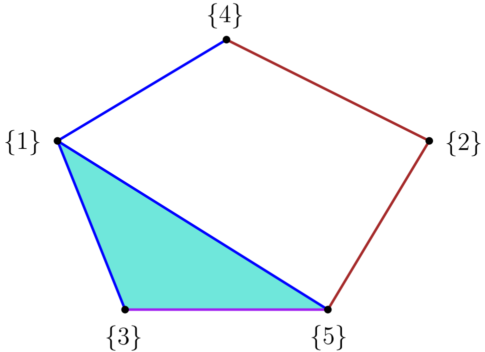

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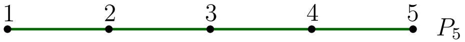


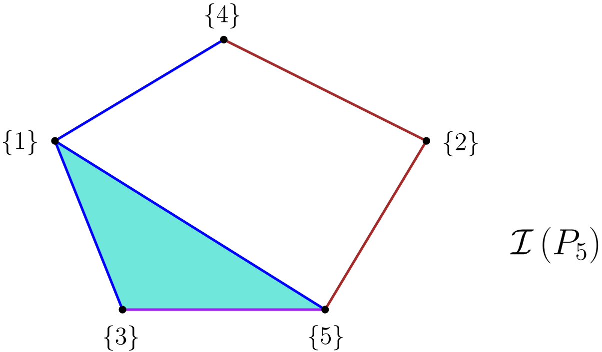

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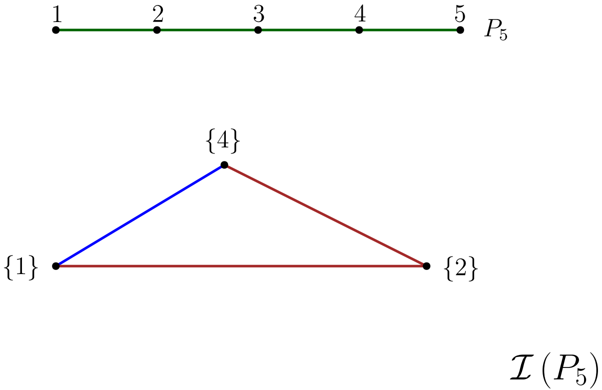

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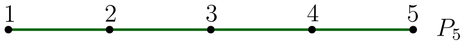


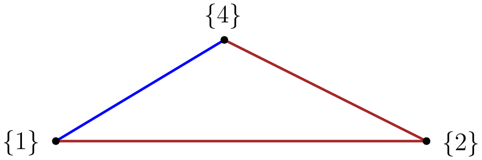


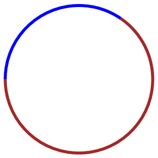

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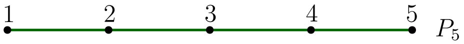


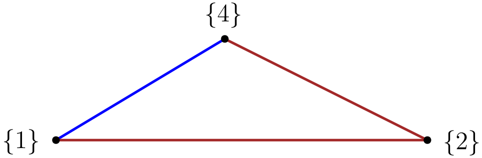


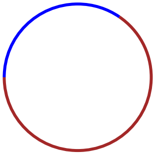

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**Theorem ([2, Proposition 4.6])**

*Let $P_m$ be the path graph on $m$ vertices. Then*

$$\mathcal{I}(P_m) \simeq \begin{cases} \mathbb{S}^{k-1}, & \text{if } m = 3k \\ pt, & \text{if } m = 3k+1 \\ \mathbb{S}^{k}, & \text{if } m = 3k+2. \end{cases}$$

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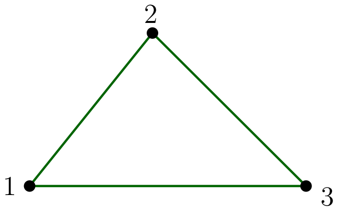

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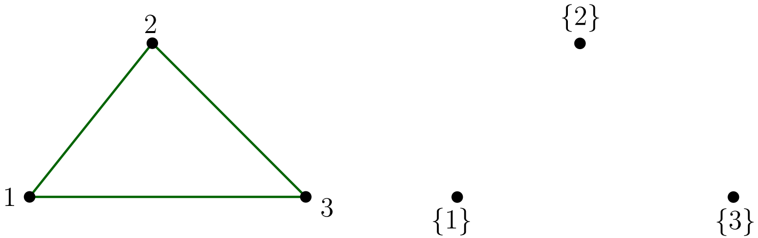

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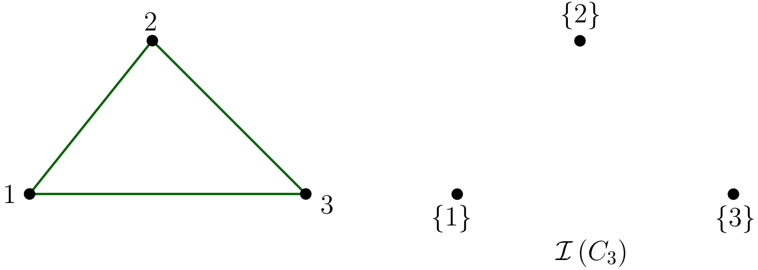

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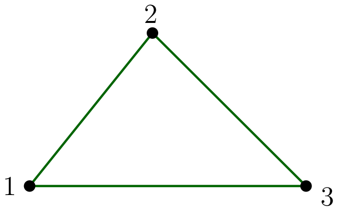


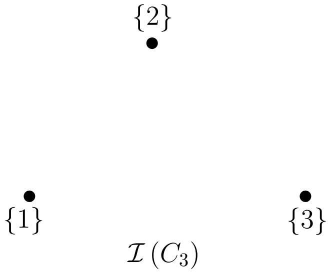


$$\mathcal{I}(C_3) = \mathbb{S}^0 \vee \mathbb{S}^0$$

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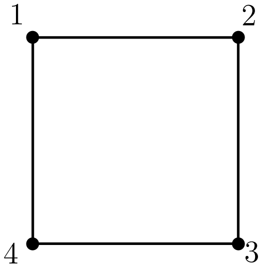

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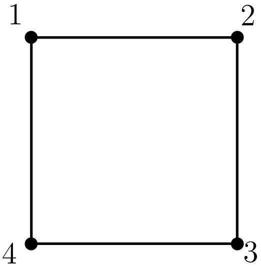


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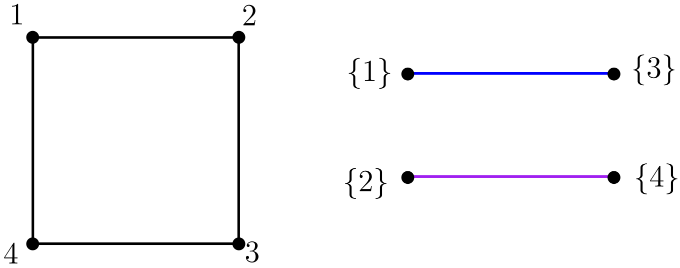

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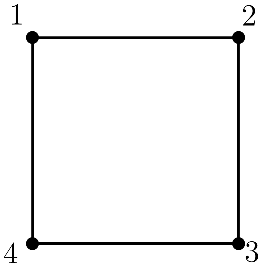


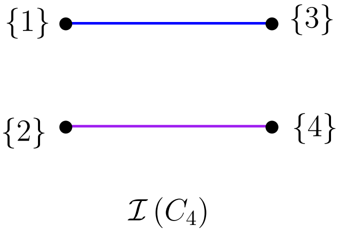

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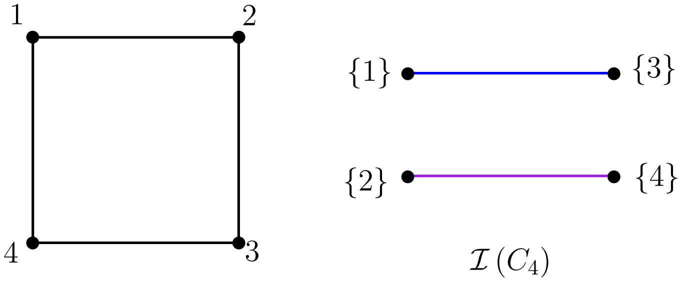


$$\mathcal{I}(C_4) \simeq \mathbb{S}^0$$

# Examples: $C_5$

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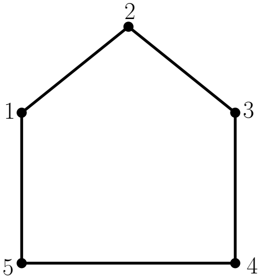
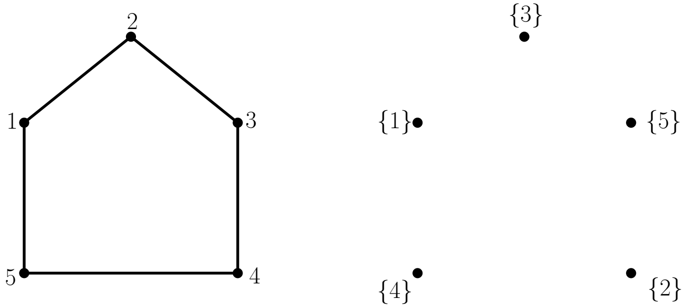


# Examples: $C_5$



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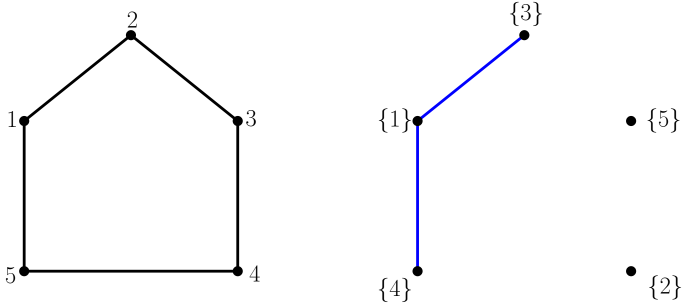

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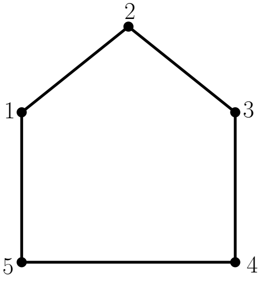


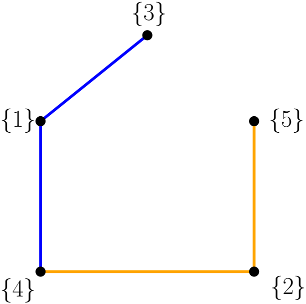

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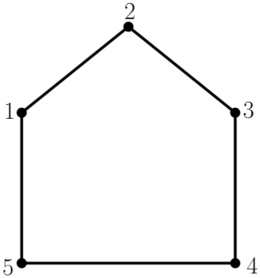


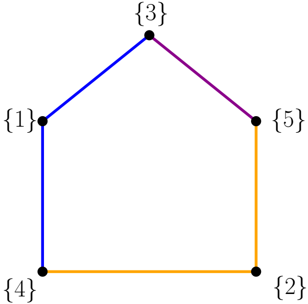

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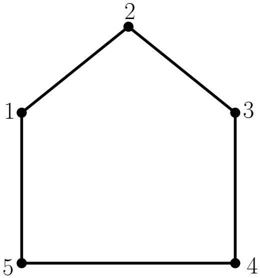


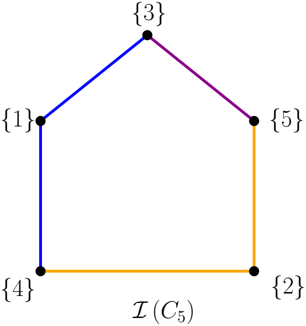

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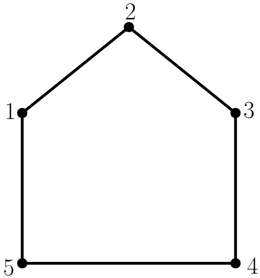
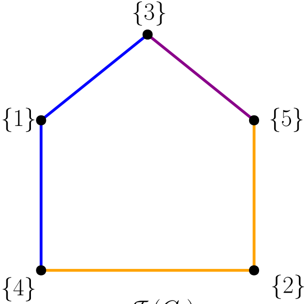
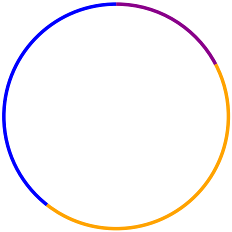



$\mathcal{I}(C_5)$



$\mathcal{I}(C_5) = \mathbb{S}^1$

# Independence complex of cycle graph

**Theorem ([2, Proposition 5.2])**

*Let $C_n$ be the cycle graph on $n$ vertices. Then*

$$\mathcal{I}(C_n) \simeq \begin{cases} \mathbb{S}^{k-1} \vee \mathbb{S}^{k-1}, & \text{if } n = 3k \\ \mathbb{S}^{k-1}, & \text{if } n = 3k+1 \\ \mathbb{S}^{k}, & \text{if } n = 3k+2. \end{cases}$$

# Main results

# Independence complex of wedge of paths

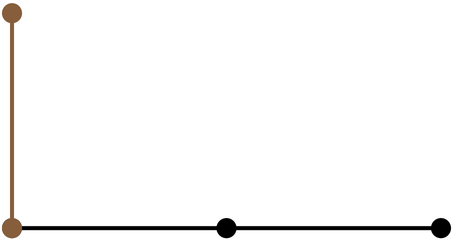
# Independence complex of wedge of paths

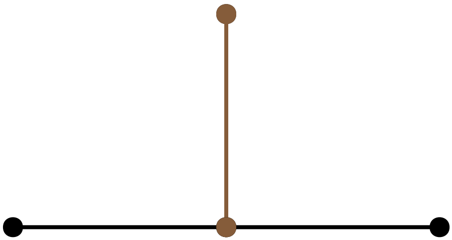
**Theorem (___, Panja, Daundkar, [1])**

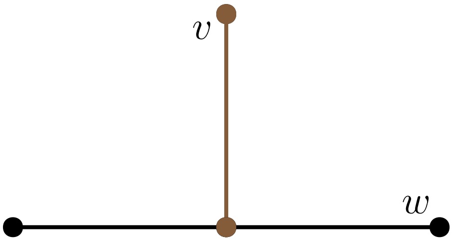
*Let $P_l$ be the path graph on $l$ vertices. Then*

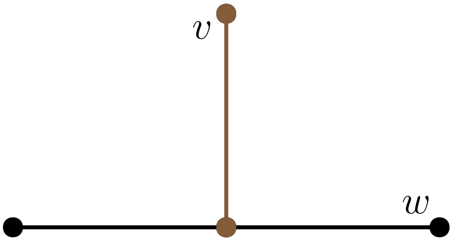
$$\mathcal{I}\left(P_m \bigvee_a P_n\right)$$

*is either a point or a sphere.*





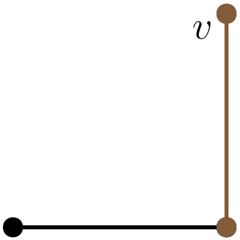

$v$
$w$




**Lemma**

*Let $G$ be a graph, and $v \neq w$ vertices of $G$. Let $N(v)$ denotes the set of all vertices $v'$ such that there is an edge between $v$ and $v'$. If $N(v) \subseteq N(w)$ for some $v$ and $w$, then*

$$\mathcal{I}(G \setminus w) \simeq \mathcal{I}(G).$$

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$$\mathcal{I}(G \setminus w) \simeq \mathcal{I}(G).$$

# Independence complex of wedge of cycles

# Independence complex of wedge of cycles

**Theorem (___, Panja, Daundkar, [1])**

*Let $C_m$ be the cycle graph with $m$ vertices. Then*

$$\mathcal{I}\left(C_m \bigvee_a C_n\right)$$

*is contractible or homotopy equivalent to wedge of spheres.*

# Independence complex of wedge of cycles

# Independence complex of wedge of cycles

## Definition (Link)

Let $K$ be a (abstract) simplicial complex. The link of a vertex $v \in K$ is defined as

$$\mathrm{lk}(v, K) := \{\sigma \in K \mid v \notin \sigma \text{ and } \sigma \cup v \in K\}.$$

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## Lemma

*Let $K$ be a simplicial complex and $v \in K$ be a vertex such that $\mathrm{lk}(v, K)$ is contractible in $\mathrm{del}(v, K)$. Then $K \simeq \mathrm{del}(v, K) \vee \sum \mathrm{lk}(v, K)$.*

# Independence complex of wedge of paths and cycles

# Independence complex of wedge of paths and cycles

**Theorem (___, Panja, Daundkar, [1])**

*The independence complex*

$$\mathcal{I}\left(C_m \bigvee_a P_n\right)$$

*is homotopy equivalent to either a point or wedge of spheres.*

# References I

N. Daundkar, S. Panja, and S. Prasad, *Independence complexes of wedge of graphs*.
https://arxiv.org/abs/2303.08798.

D. N. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A, 88 (1999), pp. 112–122.

# Thank you for your attention!