Geodesic Equation by Euler Lagrange's Equations

1 Introduction

In our whole discussion below, we assume $(\mathscr{M}, g)$ is a Riemannian manifold and all the maps are smooth (C^∞). Let $\gamma : [a, b] \to \mathscr{M}$ be a smooth curve. Recall that the *energy* functional

$$E(\gamma) := \frac{1}{2} \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|_{\gamma(t)}^2 dt. \tag{1}$$

This functional measures the total kinetic energy of a particle traveling along γ with the speed dictated by γ. In this article, we will try to find the critical point of the above functional and will show that the curve which is a critical point is a *geodesic*. Recall that if we try to find the stationary point of a function $f : \mathbb{R} \to \mathbb{R}$, we usually find x_0 such that $\left.\frac{df}{dx}\right|_{x_0} = 0$. In order to find stationary function of a functional $I(f)$ (function of functions) we usually solve a specific system of differential equation, *Euler-Lagrange Equations*. This problem is solved using the technique called *Calculus of Variations*. In the next section we will derive the Euler Lagrange equations.

2 Euler-Lagrange Equations

The calculus of variations is concerned with the maxima or minima of functionals (mappings from a set of functions to the real numbers). Calculus of variations seeks to find $y = f(x)$ such that the integral

$$I(f) = \int_{x_1}^{x_2} F(x, y, y')\, dx \tag{2}$$

is stationary. Functions that maximize or minimize functionals may be found using the *Euler–Lagrange equation* of the calculus of variations which we are going to derive. We will assume all the functions are smooth (although only C^2 is enough).

Derivation of E-L Equation. Let $y(x)$ makes I stationary and satisfies the conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

- Introduce a function $\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$
- Define
$$\bar{y}(x) = y(x) + \varepsilon\eta(x).$$
Note that $\bar{y}$ represent family of curves after fixing $\eta(x)$.

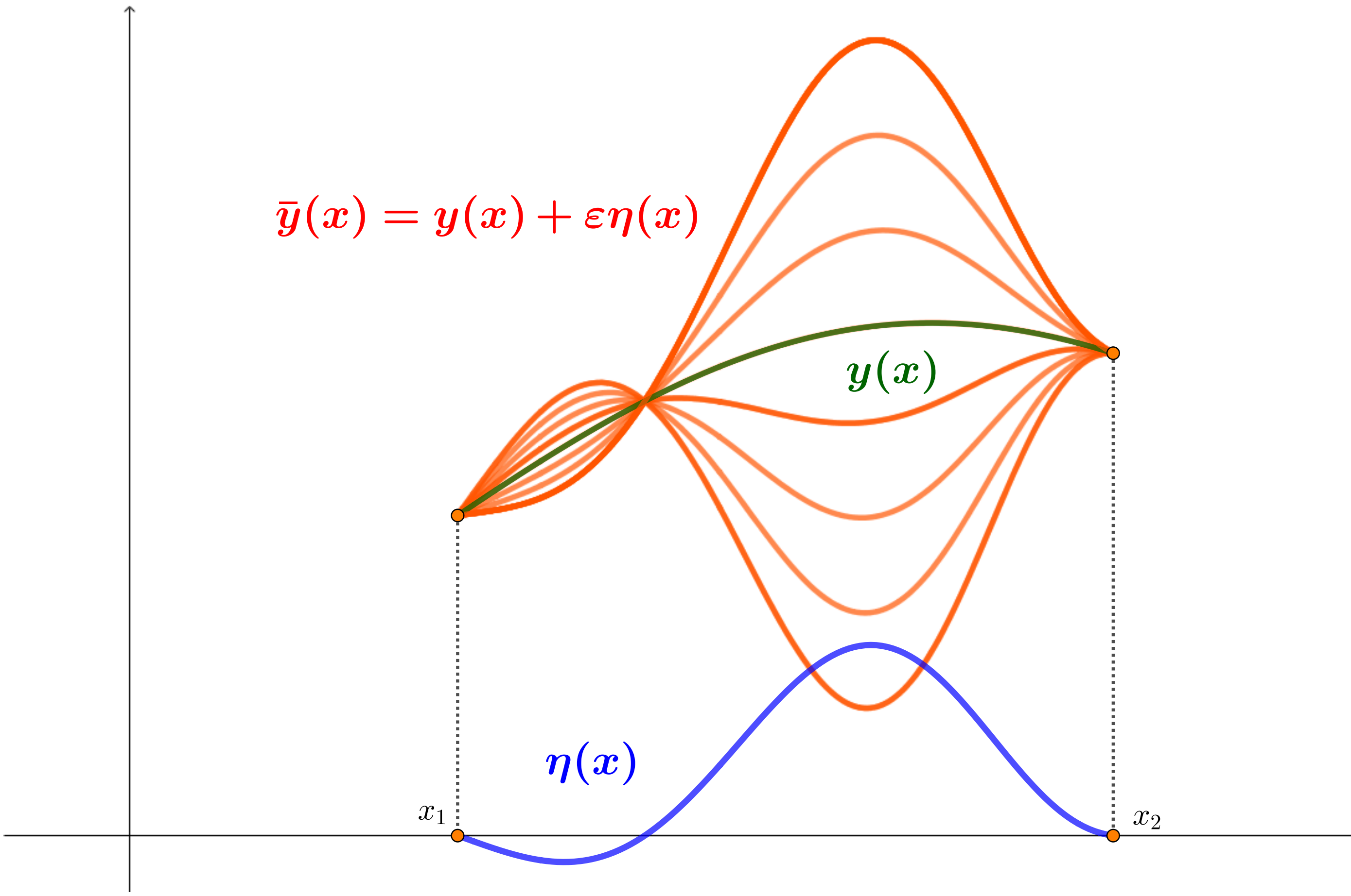

Figure 1: $\bar{y}(x) = y(x) = \varepsilon\eta(x)$

Now we can restate the problem find the particular $\bar{y}(x)$ which makes $I = \displaystyle\int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx$ stationary.
Note that I is a function of ε so to make I stationary, set

$$
\begin{aligned}
&\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0 \\
\implies &\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx\right) = 0 \\
\implies &\int_{x_1}^{x_2} \left.\frac{\partial F(x, \bar{y}, \bar{y}')}{\partial \varepsilon}\right|_{\varepsilon=0} dx = 0 \\
\implies &\int_{x_1}^{x_2} \left.\left[\frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \varepsilon} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \varepsilon}\right]\right|_{\varepsilon=0} dx = 0 \\
\implies &\int_{x_1}^{x_2} \left.\left[\frac{\partial F}{\partial \bar{y}}\eta + \frac{\partial F}{\partial \bar{y}'}\eta'\right]\right|_{\varepsilon=0} dx = 0
\end{aligned}
$$

$$\implies \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}}\eta + \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right)\right]\bigg|_{\varepsilon=0}$$

Integration by parts on the second term

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}'}\eta' \, dx = \cancelto{0}{\frac{\partial F}{\partial \bar{y}'}\eta\bigg|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right)\eta \, dx$$

$$\implies \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}} - \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right)\right)\eta\bigg|_{\varepsilon=0} dx = 0$$

$$\implies \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\eta \, dx = 0$$

The above expression is true for all η satisfying $\eta(x_1) = 0 = \eta(x_2)$ and hence

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

□

Remark. Like in the case for a function $f : \mathbb{R} \to \mathbb{R}$, the zeros of the first derivative of f are critical points, they might not be extrema, similarly the Euler-Lagrange equation is a necessary condition for a minima or maxima. It does not tell anything about the nature of the critical function.

More generally, the Euler-Lagrange equations of a functional

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t)) \, dt$$

are given by

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0, \;\; i = 1, 2, \cdots, n$$

3 Geodesic Equation

Now we have a tool to find out the critical values of functionals. So we will apply this to the energy functional (1) to get the critical points.

If in the local coordinates, the curve $\gamma(t)$ is $(x^1(\gamma(t)), \cdots, x^n(\gamma(t)))$ and we use the abbreviation

$$\dot{x}^i(t) = \frac{d}{dt}(x^i(\gamma(t))).$$

Then,

$$E(\gamma) = \frac{1}{2}\int_a^b \sum_{i,j} g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t) \, dt \tag{3}$$

Theorem. *The Euler-Lagrange equations for the energy E are*

$$\ddot{x}^i + \sum_{j,k} \Gamma^i_{jk}\dot{x}^i(t)\dot{x}^j(t)\dot{x}^k(t) = 0, \;\; i = 1, \cdots, n \tag{4}$$

with

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}\left(g_{jl,k} + g_{kl,j} - g_{jk,l}\right), \tag{5}$$

where

$$\left(g^{ij}\right)_{i,j=1,\cdots,n} = (g_{ij})^{-1} \quad \left(i.e. \sum_l g^{il} g_{lj} = \delta_{ij}\right) \tag{6}$$

and

$$g_{jl,k} = \frac{\partial g_{jl}}{\partial x^k}. \tag{7}$$

The expression Γ_i^{jk} are called Christoffel symbols.

Proof. Applying the Euler-Lagrange equations to the energy functional,

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}^i}\left(\sum_{j,k} g)jk(x(\gamma(t)))\dot{x}^j(t)\dot{x}^k(t)\right)\right] - \frac{\partial}{\partial x^i}\left(\sum_{j,k} g_{jk}(x(\gamma(t)))\dot{x}^j(t)\dot{x}^k(t)\right) = 0,\ i = 1,\cdots,n \\
\implies &\frac{d}{dt}\left(\sum_k g_{ik}\dot{x}^k + \sum_j g_{ji}\dot{x}^j\right) - \sum_{j,k} g_{jk,i}\dot{x}^j\dot{x}^k = 0,\ i = 1,\cdots,n \\
\implies &\sum_{k,l} g_{ik,l}\dot{x}^l\dot{x}^k + \sum_k g_{ik}\ddot{x}^k + \sum_{j,l} g_{ji,l}\dot{x}^l\dot{x}^j + \sum_j g_{ji}\ddot{x}^j - \sum_{j,k} g_{jk,i}\dot{x}^i\dot{x}^k = 0\ i = 1,\cdots,n \\
\implies &\left(\sum_j g_{ij}\ddot{x}^j + \sum_j g_{ji}\ddot{x}^j\right) + \left(\sum_{k,l} g_{ik,l}\dot{x}^l\dot{x}^k + \sum_{j,l} g_{ji,l}\dot{x}^l\dot{x}^j - \sum_{k,j} g_{jk,i}\dot{x}^j\dot{x}^k\right) = 0,\ i = 1,\cdots,n \\
\implies &2\sum_j g_{ij}\ddot{x}^j + \left[\sum_{j,k}(g_{ij,k} + g_{ki,j} - g_{jk,i})\right]\dot{x}^j\dot{x}^k = 0,\ i = 1,\cdots,n \\
\implies &\sum_j g_{ij}\ddot{x}^j + \frac{1}{2}\left[\sum_{j,k}(g_{ij,k} + g_{ki,j} - g_{jk,i})\right]\dot{x}^j\dot{x}^k = 0,\ i = 1,\cdots,n \\
\implies &\sum_{i,j} g^{li}g_{ij}\ddot{x}^j + \left[\sum_{j,k}\underbrace{\left(\frac{1}{2}\sum_i g^{li}(g_{ij,k} + g_{ki,j} - g_{jk,i})\right)}_{\Gamma^l_{jk}}\right]\dot{x}^j\dot{x}^k = 0 \\
\implies &\sum_j \delta_{lj}\ddot{x}^j + \Gamma^l_{jk}\dot{x}^j\dot{x}^k = 0,\ l = 1,2,\cdots,n \\
\implies &\ddot{x}^l + \Gamma^l_{jk}\dot{x}^j\dot{x}^k = 0,\ l = 1,2,\cdots,n.
\end{aligned}$$

$$\boxed{\ddot{x}^l + \Gamma^l_{jk}\dot{x}^j\dot{x}^k = 0, \quad l = 1,2,\cdots,n}$$

Thus, we obtain (4). □

We define a *geodesic* as a smooth curve $\gamma : [a,b] \to \mathscr{M}$ satisfying (4).